Time evolution of system with a time-dependent Hamiltonian: The Landau-Zener probability

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Time-dep Schrödinger equation:

$$i\hbar \frac{d}{dt}\left|\Psi(t)\right\rangle = \hat{H}(t)\left|\Psi(t)\right\rangle$$

Time-dep
Hamiltonian

Time evolution operator:

$$\left|\Psi(t)\right\rangle = \hat{U}(t)\left|\Psi(0)\right\rangle$$

System state at time t

Initial system state

- Probability amplitude for a transition from an initial state

$$\left|\Psi(0)\right\rangle$$

to a final state

$$\left|\Psi'\right\rangle$$

at time t:

$$\left\langle\Psi'\middle|\Psi(t)\right\rangle = \left\langle\Psi'\middle|\hat{U}(t)\middle|\Psi(0)\right\rangle$$

- Transition probability:

$$P_{\Psi(0)\to\Psi'}(t) = \left|\left\langle\Psi'\middle|\hat{U}(t)\middle|\Psi(0)\right\rangle\right|^2$$

The time evolution operator satisfes the eq:

$$i\hbar \frac{d}{dt}\hat{U}\left(t\right)=\hat{H}\left(t\right)\hat{U}\left(t\right)$$

It is not equal to the time evolution operator for the time-indep Hamiltonian case:

$$\xcancel{\hat{U}\left(t\right)=\theta\left(t\right)e^{-i\hat{H}t/\hbar}\equiv\theta\left(t\right)\sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^{n}\left(\frac{t^{n}}{n!}\right)\hat{H}^{n}}$$

There are no stationary states as in the time-indep Hamiltonian case.

Hence the time evolution operator cannot be written as:

$$\cancel{\hat{U}(t) = \theta(t) \sum_k e^{-iE_k t/\hbar} |\psi_k\rangle\langle\psi_k|}$$

For the time dep Hamiltonian case we can define instantaneous (adiabatic) eigenstates/eigenenergies:

$$|\psi_k(t)\rangle,\ E_k(t) \longrightarrow \hat{H}(t)|\psi_k(t)\rangle = E_k(t)|\psi_k(t)\rangle$$

Instantaneous eigenstates are non-stationary

If at t=0 the initial state coincides with an instantaneous eigenstate at t=0,

$$\left|\Psi(0)\right\rangle = \left|\psi_{k'}(0)\right\rangle$$

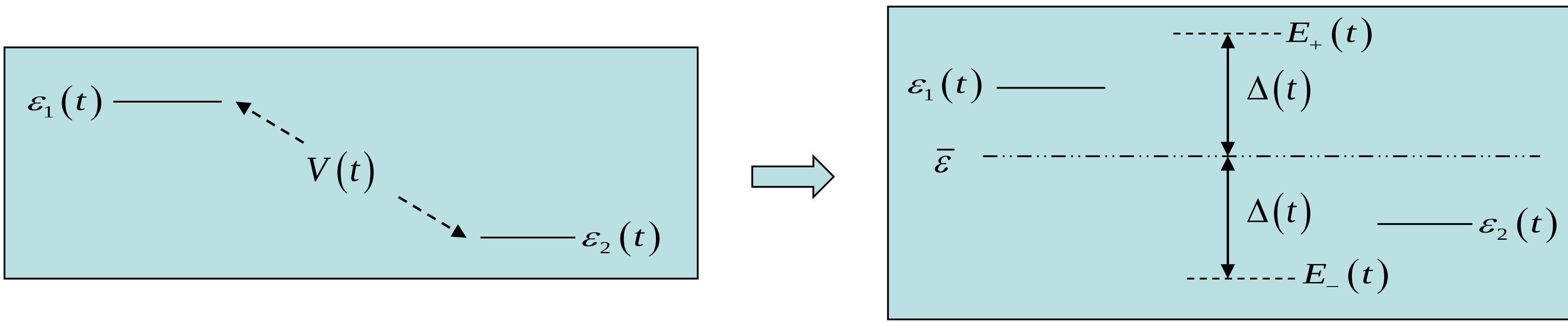
the probability to be at a different instantaneous eigenstate at a later time t is non-zero:

$$P_{\Psi_{k'}(0)\to\Psi_k(t)}(t) \neq 0$$

2x2 system example
Instantaneous (adiabatic) eigenstates/eigenenergies

If at time t there is off-resonance, e.g.,

$$\varepsilon_1(t) > \varepsilon_2(t),\ V(t) = -|V(t)|$$



$$E_\pm(t) = \underbrace{\left(\frac{\varepsilon_1(t)+\varepsilon_2(t)}{2}\right)}_{\bar{\varepsilon}} \pm \Delta(t)$$

$$\Delta(t) = \frac{1}{2}\sqrt{\left(\varepsilon_1(t)-\varepsilon_2(t)\right)^2 + \left(2|V(t)|\right)^2}$$

$$|\psi_-(t)\rangle = \frac{1}{\sqrt{2}}\left(D_-(t)|\varphi_1\rangle + D_+(t)|\varphi_2\rangle\right)$$

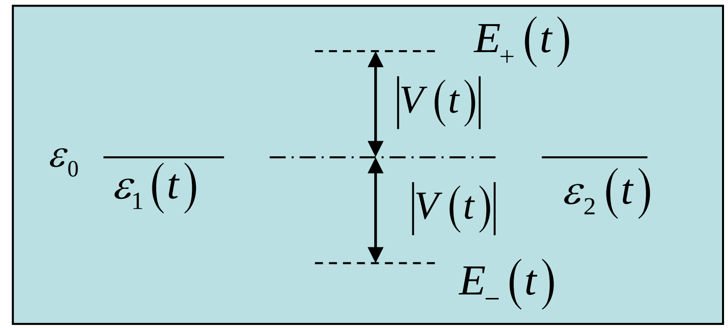
$$|\psi_+(t)\rangle = \frac{1}{\sqrt{2}}\left(-D_+(t)|\varphi_1\rangle + D_-(t)|\varphi_2\rangle\right)$$

$$D_\pm(t) = \sqrt{1 \pm \left(\frac{\varepsilon_1(t)-\varepsilon_2(t)}{2\Delta(t)}\right)}$$

If at time t there is resonance,

$$\varepsilon_1(t) = \varepsilon_2(t) = \varepsilon_0$$

$$\tilde{H} = \begin{pmatrix} \varepsilon_0 & -|V(t)| \\ -|V(t)| & \varepsilon_0 \end{pmatrix}$$



Instantaneous eigenstates/eigenenergies:

$$E_\pm(t) = \varepsilon_0 \pm |V(t)|$$

$$|\psi_-(t)\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_1\rangle + |\varphi_2\rangle\right) \qquad |\psi_+(t)\rangle = \frac{1}{\sqrt{2}}\left(-|\varphi_1\rangle + |\varphi_2\rangle\right)$$

The transition probability $\varphi_1 \rightarrow \varphi_2$ for the time dep 2x2 system is not given by the known equations of the time indep 2x2 system:

<u>Off-resonant</u>	$\varepsilon_1 \neq \varepsilon_2$	$P_{\varphi_1 \to \varphi_2}(t) = \left(\frac{\lvert V \rvert}{\Delta} \right)^2 \sin^2\left(\frac{\Delta}{\hbar} t \right)$
<u>Resonant</u>	$\varepsilon_1 = \varepsilon_2$	$P_{\varphi_1 \to \varphi_2}(t) = \sin^2\left(\frac{\lvert V \rvert}{\hbar} t \right)$

The Landau-Zener transition probability

Consider a 2x2 system with a time-dep Hamiltonian where the energies of the two states φ_1 and φ_2 change with respect to time with constant rates (approaching each other), such that they cross (states become resonant) at a time t_{res} .

We will assume that the coupling between the states is time indep.

$$\varepsilon_1(t) = \underbrace{\varepsilon_1(t_{res})}_{\varepsilon_0} + \underbrace{\left[\frac{d\varepsilon_1(t)}{dt}\right]_{t=t_{res}}}_{const\ rate} \times (t - t_{res})$$

Time-indep coupling

$$\tilde{H}(t) = \begin{pmatrix} \varepsilon_1(t) & V \\ V & \varepsilon_2(t) \end{pmatrix}$$

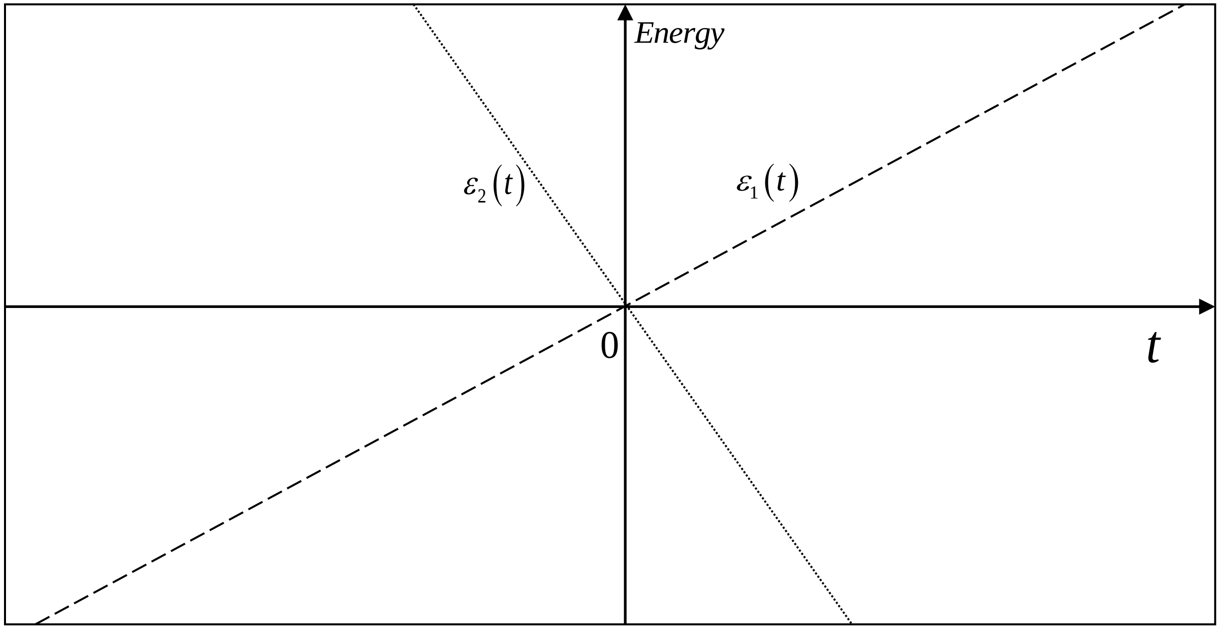
$$\varepsilon_2(t) = \underbrace{\varepsilon_2(t_{res})}_{\varepsilon_0} + \underbrace{\left[\frac{d\varepsilon_2(t)}{dt}\right]_{t=t_{res}}}_{const\ rate} \times (t - t_{res})$$

State ener at resonance

We will set the time of resonance (energy crossing time) equal to zero and the state energy at resonance equal to zero:

$$\varepsilon_0 = 0, \quad t_{res} = 0$$

$$\tilde{H}(t) = \begin{pmatrix} \left[\dfrac{d\varepsilon_1(0)}{dt}\right]t & V \\ V & \left[\dfrac{d\varepsilon_2(0)}{dt}\right]t \end{pmatrix}$$



If the initial state of the system is:

$$\left|\Psi\left(t=-\infty\right)\right\rangle = \left|\varphi_1\right\rangle$$

the transition probabily to state φ_2 after infinite time $\left(t=+\infty\right)$ is given by the Landau-Zener formula:

$$P_{1\to 2} = 1-\exp\left[-\left(2\pi\right)^2\gamma\right]$$

where γ is the Landau-Zener parameter:

$$\gamma = \frac{\left|V\right|^2}{h\left|\left(\frac{d\varepsilon_1(0)}{dt}\right)-\left(\frac{d\varepsilon_2(0)}{dt}\right)\right|}$$

Planck's constant

We write the Landau-Zener constant as a function of two time scales:

$$\gamma = \frac{\tau_{LZ}}{\tau_{rabi}}$$

The Rabi time:

$$\tau_{rabi} = \frac{h}{|V|}$$

and the Landau-Zener time:

$$\tau_{LZ} = \frac{|V|}{\left|\left(\frac{d\varepsilon_1(0)}{dt}\right) - \left(\frac{d\varepsilon_2(0)}{dt}\right)\right|}$$

Intuitive interpretation of the Landau-Zener paameter using the results for the transition probability $\varphi_1 \rightarrow \varphi_2$ In the case of a time indep 2x2 Hamiltonian

Definition of “resonance region” for time indep Hamiltonian:

Values of $\varepsilon_1,\ \varepsilon_2,\ V$ for which $|\varepsilon_1 - \varepsilon_2| \le |V|$

In this ener region, $\max\left\{P_{\varphi_1\to\varphi_2}(t)\right\} \approx 1$

Resonance $\varepsilon_1 = \varepsilon_2$ Max transition prob.= 1 $P_{\varphi_1\to\varphi_2}(t) = \sin^2\left(\frac{|V|}{\hbar}t\right)$

Off-resonance $\varepsilon_1 \ne \varepsilon_2$ Max transition prob < 1 $P_{\varphi_1\to\varphi_2}(t) = \left(\frac{|V|}{\Delta}\right)^2 \sin^2\left(\frac{\Delta}{\hbar}t\right)$

For $0 < |\varepsilon_1 - \varepsilon_2| < |V|$

Max transition prob ~ 1

$$P_{\varphi_1\to\varphi_2}(t) = \left(\frac{|V|}{\Delta}\right)^2 \sin^2\left(\frac{\Delta}{\hbar}t\right) \approx \sin^2\left(\frac{|V|}{\hbar}t\right)$$

For $|\varepsilon_1 - \varepsilon_2| >> |V|$

Max transition prob << 1

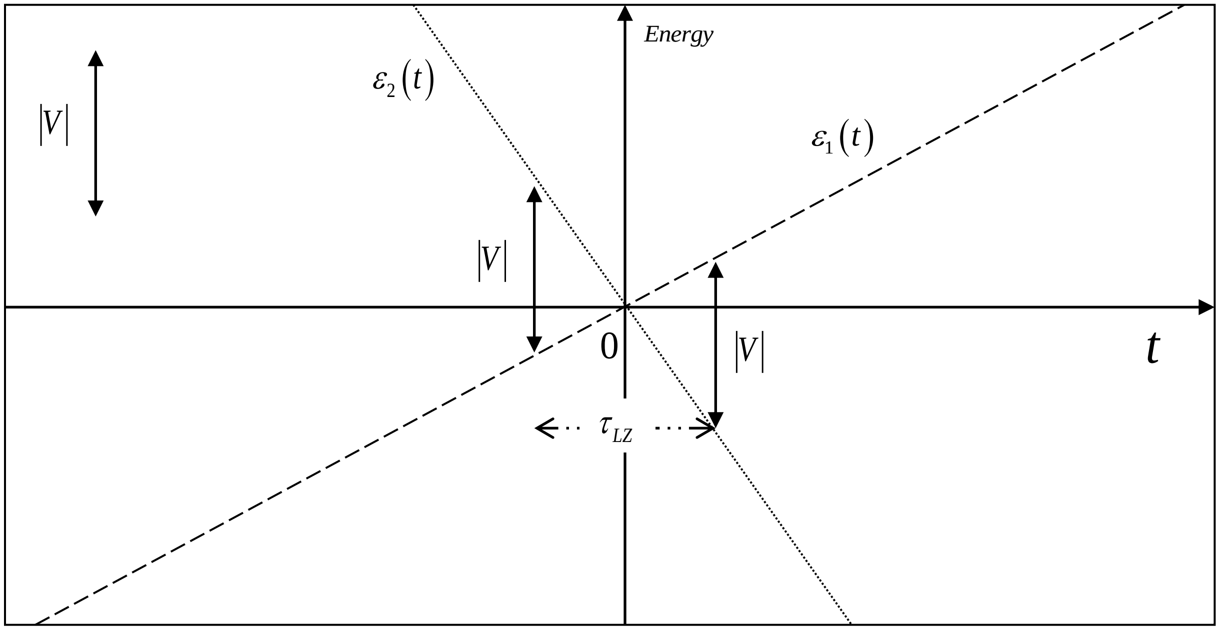
$$P_{\varphi_1\to\varphi_2}(t) = \left(\frac{|V|}{\Delta}\right)^2 \sin^2\left(\frac{\Delta}{2\hbar}t\right) \approx \left(\frac{2|V|}{\varepsilon_1 - \varepsilon_2}\right)^2 \sin^2\left(\frac{[\varepsilon_1 - \varepsilon_2]}{2\hbar}t\right)$$

Landau-Zener time

$$\tau_{LZ} = \frac{|V|}{\left| \left(\frac{d\varepsilon_1(0)}{dt} \right) - \left(\frac{d\varepsilon_2(0)}{dt} \right) \right|}$$

Approximate time that the system remains in the resonance region (as defined for time indep system):

$$|\varepsilon_1(t) - \varepsilon_2(t)| < |V(t)|$$

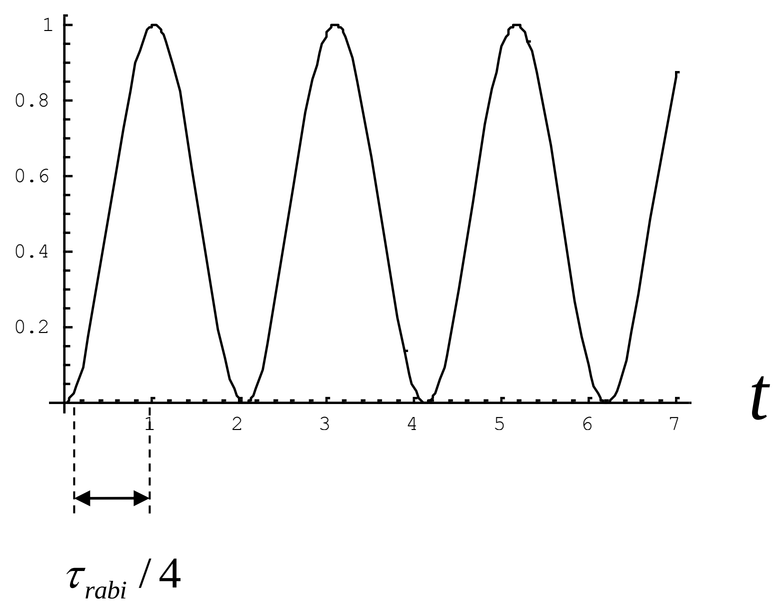


Rabi time

$$\tau_{rabi} = \frac{h}{|V|}$$

Approximate time needed for a complete transition $\varphi_1 \rightarrow \varphi_1$ (100% prob) for a resonant 2x2 time indep Hamiltonian system:

$$P_{\varphi_1 \to \varphi_2}(t) = \sin^2\left(\frac{|V|}{\hbar} t\right)$$



Nonadiabatic behaviour of Landau-Zener probability

$$\tau_{LZ} << \tau_{rabi} \;\rightarrow\; (2\pi)^2\gamma << 1$$

Small transition probability to state φ_2 after infinite time $\left(t = +\infty\right)$:

$$P_{1\rightarrow 2} = 1 - \exp\left[-\left(2\pi\right)^2 \gamma\right] \approx \left(2\pi\right)^2 \gamma << 1$$

The energies of φ_1 and φ_2 go through the resonance region very fast wrt the typical φ_1 to φ_2 transition time at resonance →

τ_{LZ} much smaller than τ_{rabi}

Adiabatic behaviour of Landau-Zener probability

$$\tau_{LZ} >> \tau_{rabi} \;\; \rightarrow \;\; (2\pi)^2\gamma >> 1$$

The transition probability to state φ_2 after infinite time $\left(t = +\infty\right)$ is 100%

$$P_{1\rightarrow 2} = 1 - \exp\left[-\left(2\pi\right)^2 \gamma\right] \approx 1$$

The energies of φ_1 and φ_2 go through the resonance region slowly wrt the typical φ_1 to φ_2 transition time at resonance →

τ_{LZ} much larger than τ_{rabi}

The Landau-Zener parameter in the case where the energies of φ_1 and φ_2 depend on time through an external parameter R(t):

$$\varepsilon_1(t) = \varepsilon_1\left[R(t)\right]$$

$$\varepsilon_2(t) = \varepsilon_2\left[R(t)\right]$$

$$\gamma = \frac{|V|^2}{h\left|\left(\frac{dR(t)}{dt}\right)_{t=0}\right| \times \left|\left(\frac{d\varepsilon_1(R)}{dR}\right)_{R=R(0)} - \left(\frac{d\varepsilon_2(R)}{dR}\right)_{R=R(0)}\right|}$$