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DOTTORE DI RICERCA

FAILURE OF DISORDERED MEDIA

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The understanding of the properties of fractures is a problem rooted in time, since the reliability of materials has become of some interest for humanity. At the present the understanding of the structure of materials is well established, at least at lengthscales where fractures come into play. Notwithstanding, there are many aspects of fracture mechanics, such as size effects and scaling properties, that are still unsolved problem. Fracture is essentially a multidisciplinary phenomenon, spanning through many lengthscales, each of these related to a different fields of science. From the study of the dynamic at the atomic level, to the construction of huge structures, many frameworks in science and engineering come into play. We move into one of the most recent, the statistical mechanics of fracture. Far enough from the basics constituents of matter to forget their complex internal structure, but still able to be sensible to microscopic characteristics like the size of the defects.

The first formulations of a statistical theory of fractures date back to the works of Griffith [1] and Weibull [2]. Despite this early start, the main approaches have been mostly from an experimental point of view: experiments on real materials has been the preferred way to discover the constitutive laws that rule the breakdown of materials with respect to a more theoretical approach. But in the last decades a statistical approach becom compelling, due to the emergence of problems that are familiar to statistical physics. For instance, size-effects are rooted in disorder, and different samples of the same material generally display fluctuations in the strength. The importance of defects and fluctuations in fracture can be well formalized from a statistical point of view. In addiction, the statistical description of failure can also be cast in terms of the renormalization group since one is interested in the rescaling of a whole function (strength distribution) and its fixed point limit.

Fracture can be seen as the outcome of the irreversible dynamics of a long-range interacting, disordered system. Several experimental observations have revealed that fracture is a complex phenomenon, described by scale invariant laws [3]. The observation of scaling makes it attractive to discuss breakdown processes in terms of statistical mechanics and in particular to search for analogies with phase transitions and critical phenomena. For these analogies, the statistical
properties of fracture have attracted a wide interest in the statistical physics community \[4-14\].

The statistical physics of fracture models are usually investigated by means of discrete lattice models in which the elastic medium is replaced by a network of discrete elements. The cornerstone in this respect is the Random Fuse Model (RFM) \[7,15\], a lattice model in which as a further key simplification vectorial elasticity has been substituted with a scalar field, but many other models, each referring to a particular choice of the physics to represents, has been proposed. From exactly soluble fiber models to more complex tensorial models, these models describe on a qualitative level the failure processes of real, brittle, or quasi-brittle materials. This has been particularly important in solving the classical engineering problems of material strength: the size dependence of maximum stress and its sample-to-sample statistical fluctuations. At the same time, lattice models pose many new fundamental questions in statistical physics, among these partly still controversial issues, are the scaling and size-effects in material strength and accumulated damage, and the relation between fracture and phase transitions.

The theoretical concepts needed to understand the distribution of fracture strength rely on the statistics of extremes \[4,16\]. The main advance in this field was made in the 1980’s by Duxbury and coworkers \[17\]. While the Weibull distribution is still in extensive use in materials science though as outlined its theoretical basis is under debate, Duxbury and coworkers presented some evidence for the superiority of the modified Gumbel theory.

In this work we intend to clarify some aspects of the scaling of the strength distribution and the related size-effects. By the means of massive numerical simulations of the diluted disorder random fuse model we was able to reach a statistical confidence never reached before in this field of research. This allow us to obtain a clear insight into the factors that drive the fracture process, especially the strenght distribution of the system and the role of the width distribution of the clusters are taken into account. The scaling analysis of the observables is given for all the main steps which divides the loading process, namely the first bond breaking, the damage accumulation up to the peak load, the peak load and the postfailure regime. A characterization of the dynamical growth of the spanning crack in the intermediate steps is also carefully outlined. The results of the investigation partly confirm the theoretical framework derived from the basic assumption of the theory of the linear elasticity, but at the same time we unveiled some new behaviour that need to be framed in a most general theory. Especially the data show a strong statements in favor of the Gumbel distribution as the leading distribution for the weakly disordered rfM.

This thesis is organized as follows. In chapter 2 we give a brief introduction to basic notions needed to understand the fracture mechanics; we start from the theory of linear elasticity along with the most common nonlinear behaviour, and we introduce the brittle and ductile fractures. Then we proceed with the theories about the fracture mechanics, the Griffith’s criterion and the
energetics of microcracks. We end presenting the important concept of damage and disorder. The most common fracture models are introduced in chapter 5. We first review the various theoretical approach to the field, then we present in more details some specific model relevant for the statistical mechanics, such as the fiber models and the network models. In chapter 4 are treated in detail the statistical aspects of fracture, in particular with relation to the ways they are introduced in the models.

The details about our model are presented in chapter 5 and we give a detailed analysis of statistical fracture in the random fuse model in following chapters 6, 7 and 8 are devoted to the analysis of the random fuse model simulations.
Linear elastic fracture mechanics (LEFM) describes the formation and propagation of cracks in terms of macroscopic field equations, under the assumptions of the theory of linear elasticity. The mechanical behaviour of a flawless, homogeneous medium is the basis of the understanding of the breaking of a solid. At an atomistic level, the behaviour of a material can be modeled accounting for all the forces acting on each atom or molecule, but on length scales on which one does not feel anymore the chemistry or the texture of the material its mechanical behaviour is determined by a few quantities which enter in what are called the constitutive relations. These relations (as the Hooke’s law) tell us how a material reacts to a local application of a force.

2.1 Linear Elasticity

The theory of linear elasticity (LE) describes how solid bodies deform under the action of external forces. In the framework of continuum mechanics, the theory is a simplification of the nonlinear theory of elasticity in which the materials are modeled as continuous. The assumptions that yield to the linear theory are that, under the application of a load,

- the deformation of the body is small;
- it exists a linear relationship between the applied stress and the strain.

Under these assumptions, the behaviour of many engineering materials and design scenarios are well described.

2.1.1 Young’s modulus and Poisson ratio

Let us now introduce the classical experiment by Young. Consider an homogeneous bar of length $L$ with cross section $S = w^2$ and apply a force $F$ along the axis. Due to the load the block is elongated by $\Delta L$ and its width decreased by $\Delta w$. The relative variations $\delta = \Delta L/L$ and $\omega = \Delta w/w$ can be measured as a function of $F$. 
In the regime of small elongations one finds that the relation between \( F \) and \( \delta \) is usually linear and reversible: then we can apply the formalism of linear elasticity. In the theory, an isotropic medium is characterized by two elastic moduli, the Young’s modulus \( E \) and the Poisson’s ratio \( \nu \), defined respectively by

\[
E = \frac{F S}{L \Delta L} \quad (2.1)
\]

\[
\nu = -\frac{\Delta w}{w} \frac{\Delta L}{L} \quad (2.2)
\]

where the physical range of these moduli is \( E \geq 0 \) and \(-1 \leq \nu \leq \frac{1}{2} \).

### 2.1.2 Constitutive relations

We parameterize now the solid by a set of coordinates \( r_0 \) assuming again that it belong to the range of validity of the LE framework, so there exists a direct relation between the coordinates \( r_1 \) of the deformed solid with those \( r_0 \) of the undeformed solid. We define the displacement field as

\[
u = r_1 - r_0. \quad (2.3)
\]

The force acting on an infinitesimal part \( dV \) of the material can be seperated in body external forces acting over the volume, body internal forces (that cancel due to the action-reaction principle) and surface forces, due to the applied traction, acting on the surface \( dS \). We can write the external force vector as function of the body force per unit volume \( b \) and of the force per unit surface area \( t \). Using the Cauchy’ hypothesis, \( t = n \cdot \sigma \), we can express the traction as function of the unit outward normal \( n \) to the surface \( dS \) and the Cauchy stress tensor \( \sigma \), obtainning that the external force vector is

\[
F = \int_V b dV + \int_S t dS = \int_V b dV + \int_S n \cdot \sigma dS. \quad (2.4)
\]

Applying the divergence theorem, the previous equation becomes

\[
F = \int_V (b + \nabla \cdot \sigma) dV, \quad (2.5)
\]

or

\[
F_i = \int_V dV \left( b_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \right)
\]

in component form.

The motion equation for the infinitesimal element \( dV \) reads

\[
b + \nabla \cdot \sigma = \rho \ddot{u}. \quad (2.6)
\]
2.1. LINEAR ELASTICITY

where \( \rho \) is the density of the material. If the body is in equilibrium the displacement is not a function of time, \( \rho \ddot{u} = 0 \):

\[
\sum_j \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0.
\]

When the body forces \( b \) are absent, equation (2.7) reduces to the equilibrium condition

\[
\sum_j \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \text{or } \nabla \cdot \sigma = 0.
\]

Let us measure the displacement of the deformed solid relative to the solid at rest defining the symmetric strain tensor through the strain-displacement equation

\[
\epsilon = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right]
\]

In the limit of small deformations, there exists a linear relation between stress and strain

\[
\sigma = C \cdot \epsilon,
\]

where \( C \) is the (fourth-order) stiffness tensor of components \( C_{ijkl} \). This is the Hooke’s law, the most general linear expression relating the strain and the stress tensor. In the case of isotropic media, the stiffness tensor has no preferred direction and the applied force give the same displacement (in the force direction) irrespective of the direction in which the force is applied.

\[
C_{ijkl} = K \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl}),
\]

where \( \delta_{ij} \) is the Kroneker delta, \( K \) is the bulk modulus and \( \mu \) is the shear modulus (Lamé second parameter).

The equation (2.10) can be rewritten, in components, as

\[
\sigma_{ij} = K \delta_{ij} \sum_k \epsilon_{kk} + 2\mu \left( \epsilon_{ij} - \frac{1}{3} \delta_{ij} \sum_k \epsilon_{kk} \right).
\]

This expression is important since it separates the stress into a scalar part associated with a scalar pressure — involving a volume change — and into a traceless part — associated with shear force which does not imply any volume change.

Usually one simplifies the expression of the stiffness tensor rewriting the Hooke’s law as function of the Lamé coefficients. Equations (2.11) and (2.12) become

\[
C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]

and

\[
\sigma_{ij} = \lambda \delta_{ij} \sum_k \epsilon_{kk} + 2\mu \epsilon_{ij}.
\]
CHAPTER 2. FRACTURE MECHANICS

Another common formulation of the Hooke’s law involves the Young’s modulus \( E = \frac{9K\mu}{(\mu+3K)(\mu+3K+2\mu)} \) and the Poisson’s ratio \( \nu = \frac{3K-2\mu}{2(3K+\mu)} \):

\[
\sigma_{ij} = \frac{E}{1-\nu} \left( \epsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} \sum_k \epsilon_{kk} \right).
\]

(2.15)

The equation of the stress equilibrium (2.8) for an isotropic elastic medium can be written in terms of the displacement field \( \mathbf{u} \) the Lamé coefficients \( \lambda \) and \( \mu \) as

\[
(\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} = 0.
\]

(2.16)

This is the Lamé equation for a homogeneous material.

2.1.3 Constitutive relations: an alternative derivation

Equation (2.14) arises naturally following a more axiomatic derivation of the theory, starting from a basic requirement. We require that the theory should be invariant under Galilean transformations (combinations of rotations and translations); this is equivalent to require that such a transformation should change neither its energy nor any physical quantity we can derive, as the stress field.

To fulfill the requirement, we express the elastic energy as a function of the derivatives of the displacement field. The gradient of \( \mathbf{u} \) is a second-order tensor whose components \( i, j \) can be written¹ as \( \partial_i u_j \). Moreover, we can split it in a symmetric and an antisymmetric part, the latter one corresponding to the rotational part. The energy must be translationally invariant, so it depends only on the symmetric part. We call this symmetric part the strain field

\[
\epsilon_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i).
\]

(2.17)

As in the previous derivation, we assume again the solid as homogeneous and anisotropic. Now we can use only invariant of the tensor \( \epsilon \) introducing an expression quadratic in the strain field. This lead to the general expression for the elastic energy \( \mathcal{E} \):

\[
\mathcal{E} = \frac{1}{2} (2\mu \mathrm{tr}(\epsilon^2) + \lambda \mathrm{tr}(\epsilon)^2),
\]

(2.18)

where \( \lambda \) and \( \mu \) are as usual the Lamé coefficients.

The stress field \( \sigma \) is introduced as the conjugate field to the strain field and it also is, by construction, a second-order symmetric tensor that is related to the strain field by the Hooke’s law

\[
\sigma_{ij} = \frac{\partial \mathcal{E}}{\partial \epsilon_{ij}}.
\]

(2.19)

¹A remark about notation: in the following we make use of the implicit summation on repeated index and we use the contraction \( \partial_i = \partial_i/\partial x_i \).
or

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk}$$ \hspace{1cm} (2.20)

The conjugate variable of a displacement is a force (the product of both being an energy). We therefore expect a relation between the stress tensor and the force distribution inside a solid. It is indeed easy to work out, for a uniform strain in a cubic solid, the force that has to be applied on one face to equilibrate the stress. For an infinitesimal surface \(dS\), where the stress field can be considered homogeneous, the force \(F_i\) is given by

$$F_i = \sigma_{ij}n_j dS, \hspace{1cm} (2.21)$$

where \(n\) is a unit vector normal to the surface \(dS\).

Galilean invariance implies that the stress field should not do any work during a rigid motion of the solid. This property can be used to derive the \textit{balance equation} satisfied by the stress field. When the solid is subjected to an external force density \(f\), the balance equations read

$$\partial_i\sigma_{ij} + f_j = 0. \hspace{1cm} (2.22)$$

Substitution of equations (2.17) and (2.20) into (2.22), allows to obtain the Lamé equation for the displacement field (the equation (2.16)):

$$(\lambda + \mu)\partial_i\partial_j u_{ij} + \mu\partial_i\partial_k u_{jk} + f_i = 0$$ \hspace{1cm} (2.23)

2.1.4 \textbf{Connections between conductivity and elasticity}

To highlight the connections between conductivity and elasticity, we derive the basic equations for this latter case. The analogue of the displacement field in a conductivity problem is the potential field. In this respect, elasticity is sometimes referred as a vector problem, opposed to the scalar case of conductivity. We postulate that the electric energy depends only on the first derivative of the potential field \(V(x)\), which is called the \textit{electric field}, \(e_i = \partial_i V\). For an isotropic solid, one obtains the following expression for the energy (cfr (2.18)):

$$\mathcal{E} = \frac{1}{2} [k(e_ie_i)], \hspace{1cm} (2.24)$$

where \(e^2 = e_ie_i\) is the only invariant and the constant \(k\) is the \textit{conductivity}. The conjugate field of the potential is the \textit{current} \(j\) (cfr (2.19))

$$j_i = \frac{\partial\mathcal{E}}{\partial e_i} = ke_i. \hspace{1cm} (2.25)$$

The \textit{flux} \(\Phi\) of charge through a surface \(dS\) is related to the current through a relation similar to (2.21):

$$\Phi = j_i n_i dS, \hspace{1cm} (2.26)$$
where \( \mathbf{n} \) is the unit vector normal to the surface. The balance equation (cfr (2.22)) becomes the conservation of charge that, in presence of a source term \( \phi \), can be written as

\[
\partial_i j_i + \phi = 0
\]  
(2.27)

Finally, expressing the previous equation in term of the potential, we obtain the Laplace equation, the conductivity equivalent of the Lamé equation (2.23):

\[
k \partial_i \partial_i V = \phi.
\]  
(2.28)

2.2 Non-linear behaviours

Up to now, we assumed that the behaviour of the material was linear and elastic. This is true for small displacement when a force \( F \) is applied, but unfortunately, this case is not always true. If the force is increased beyond a certain material-dependent value \( F_N \) one sees deviations from the linear law. In some specific cases it can even happens that the relation is non-linear already for infinitesimal forces due to force-dependent internal contacts. In these cases the elastic moduli are force-dependent functions and one calls \( k = \frac{dF}{d\Delta L} \) the “stiffness” and its inverse the compliance of the system. In the following we review briefly the most common cases where a non-linear behaviour arises.

2.2.1 plasticity

![Figure 2.1: Constitutive relations obtained through Young's experiment for copper, mild steel and the ideal case of perfect plasticity. From [4]](image)

In the elastic (or reversible) regime the system goes back to exactly the original shape when the force is reset to zero. If a certain material-dependent force \( F_Y \), called the yield point, is passed this reversibility is lost and one finds the so-called plastic behaviour. Beyond \( F_Y \) (at \( A \) in figure 2.1a), a finite, permanent elongation, called “plastic deformation”, remains when the force is reset to zero which is the point \( B \) in figure 2.1a.
Plasticity is due to flow inside the material, commonly flow along crystal planes. In the ideal case of "perfect plasticity", i.e. when $dF/d\Delta L = 0$ beyond $F_Y$ as shown in figure 2.1 and figure 2.2, the flow is indistinguishable from that of some fluids.

Figure 2.2: A perfectly plastic solid. The behaviour is elastic during OA. In A the stress reaches the yield point and during AB the behaviour is plastic. Decreasing the stress at B results again in an elastic behaviour along BC, but with a finite strain $\epsilon_p$ that remains when $\sigma$ is zero. (From [7])

Plasticity is characterized by an irreversible deformation that survives once a plastic strain has taken place, even when the forces applied onto the system are reset at zero.

In figure 2.2, for small stress, the behaviour is elastic (OA) and no irreversible deformations occur. But when a threshold stress, the yield stress $\sigma_p$, is reached at A, increasing the strain results in plastic flow, and the stress keeps its threshold value. Let us suppose that we reach point B and then decide to release the stress. The stress and strain will follow the line BC, as if the behaviour was elastic, with the same elastic modulus as in the first stage OA. However, once the stress is zero, there exists a non-zero strain, $\epsilon_p$, called plastic strain. It is irreversible. If the strain increases again from C, we will follow the line CB, up to the threshold stress where plastic flow will take place again (BD).

In linear elasticity, we have a linear relation between $\sigma$ and $\epsilon$; here, for less than the yield stress, we can have any desired strain. To characterize the state of the material we introduce another variable: we split the strain in two parts: the "plastic" strain $\epsilon_p$ that would remain if the stress was reset to zero, and the "elastic" strain $\epsilon_e$:

$$\epsilon_t = \epsilon_p + \epsilon_e.$$  \hfill (2.29)

At a given total strain many stress states could exist depending on the history of the material. Therefore, we have to introduce an incremental law, which gives a relation between time derivatives (denoted by a dot) of the strain and the stress. We distinguish two cases:

1. if $\sigma$ is less than the yield stress, $\sigma < \sigma_p$, or if $\sigma < \sigma_p$, and $\dot{\epsilon}_t < 0$, the incremental behaviour is elastic: $\dot{\epsilon}_p = 0$ and $\dot{\epsilon}_e = (1/E)\dot{\sigma}$. The total strain rate is equal to the elastic strain rate as follows from equation (2.29).
2. If \( \sigma = \sigma_p \) and \( \dot{\epsilon}_i > 0 \), the stress will remain at its threshold, \( \dot{\sigma} = 0 \), and the elastic strain \( \dot{\epsilon}_e \) will not vary. The total strain rate is equal to the plastic strain rate.

Therefore, knowing the present state, \( \sigma, \epsilon_e, \epsilon_p \), and the total strain rate we can compute the evolution of the stress and of each strain. This model is very simple if only stresses in one direction are considered.

In the majority of cases, however, one has a finite strain hardening, i.e. \( dF/d\Delta L \) is non-zero beyond \( F_Y \) as is the case in figures 2.1 and 2.1b. Strain-hardening implies that once \( F_Y \) is passed and the force is reset to zero and increased again the system becomes stiffer as can be seen from figure 2.1b: the curve BA is steeper than OA. The internal flow has modified the material and made it harder.

### 2.2.2 Viscoelasticity

Deforming a solid material, it may happen that the deformation causes energy dissipation, e.g. heating, and as a consequence the response of the system will not be immediate but delayed. Viscoelasticity is formally an extension of linear elasticity, where the stress state depends not only on the present strain, but also on the past. This is accounted generalizing the Hooke’s law (2.10) non-locally in time:

\[
\sigma_{ij} = \int_0^\infty C_{ijkl}(\tau) \epsilon_{kl}(t - \tau) d\tau. \tag{2.30}
\]

We recover elasticity as the limiting case when the delay kernel \( C \) is a Dirac distribution in time.

### 2.3 Crack mechanics

Every solid breaks if a sufficiently large load is applied on it. The value of this load as well as the shape and other characteristics of the resulting crack strongly depend on the material and on the way the load has been applied. In order to understand why and how a given sample breaks one can pursue various approaches depending essentially on the length scale in which one is interested. Ranging from atomic scales to the size of grains, the mechanisms involved in fracture are very diverse and strongly vary from one material to the other. These microscopic scales are studied by material scientists. On macroscopic scales, from centimeters upwards, fracture is a problem of engineering. In the intermediate range, between microns and centimeters, the mechanical behaviour can be described by continuum equations in which the material is characterized by some parameters and where only few types of behaviour, like the yet encountered elastic, plastic or viscoelastic, are distinguished.

The fracture of an ideal crystal can be handled theoretically and various properties can be calculated from first principles. Unfortunately, however, the force needed to break the crystal...
that one obtains from these calculations is several orders of magnitude larger than forces measured experimentally on real materials. The reason for this discrepancy is that real substances possess "disorder", i.e. they have deviations from the perfect structure and the process of fracture is extremely sensitive to disorder. Microscopically disorder can mean many things: vacancies, inclusions, composite structure, dislocations or grain boundaries but on the mesoscopic length scales, these spatial inhomogeneities can be reduced to a random noise in the material properties. This form randomness is familiar to statistical physicists. In such a formalism one can find critical phenomena, like percolation.

Figure 2.3: The three modes of loading. From [8]

For the sake of simplicity we considered in most examples uniaxial experimental setups. In real applications one has, however, seldom an axial symmetry of the load, usually both the externally applied forces and the resulting deformations of the solid are complicated three-dimensional vector fields. In this respect, it is convenient to identify three fracture modes (see figure 2.3) corresponding to tensile (I), shear (II), and tearing (III) conditions. Any arbitrary load can be expressed in a linear theory as a superposition of these three modes [18, 19].

2.4 Brittle and ductile

Coming back to Young’s experiment; when one reaches a certain force $F_c$ the sample will break apart. If this happens before getting to the yield point $F_c \leq F_Y$ then the system behaves elastically until the breaking and fracture is called brittle. In the opposite case $F_c > F_Y$ the sample breaks in the plastic regime and the fracture is called ductile. Brittle or ductile fracture are not fixed properties of the material but depend for instance on temperature or pressure. Transitions from brittle to ductile are found increasing the temperature or decreasing the speed of the deformation.

It is useful to consider the dependence between force $F$ and elongation $\delta$ which in the case of fracture is also called the “breaking characteristics”. In both cases, brittle and ductile, the characteristics follows for small forces the constitutive relation of the material. But for larger forces the material looses stiffness due to changes that occur inside the material. At $F_c$ the curve reaches its maximum and after that the sample elongates and breaks without any increase in the externally applied force $F$. The regime after the maximum is called “postfailure” regime.
In the case of brittle fracture the characteristics in the postfailure regime can show very strong variations from sample to sample. In ductile fracture very different phenomena appear, like geometrical instabilities. Let us consider a metal bar under uniaxial tension (the Young’s experiment); here an instability called “necking” can occurs: after passing the yield point flow occurs on certain crystal planes rendering the bar slightly thinner at some regions as compared to the rest (localization). Since in these regions the cross section is smaller the force per area is larger and more elongation, i.e. more flow of crystal planes will take place. Consequently the region will become thinner and thinner and the force per area will finally diverge so that the sample tears apart at this neck. Another well-known instability, called “buckling”, occurs under uniaxial compression: the axial symmetry is broken and the bar bulges into one direction.

2.5 Single crack behaviour

Real materials possess a large number of heterogeneities which can range from very localized interstitial vacancies to rather extended grain boundaries. At length scales at which the medium can be approximated by an elastic continuum we can discard the detailed nature of these heterogeneities and consider that there are randomly distributed “microcavities” in the medium. Usually one distinguishes two types of microcavities: “pores” which have the same length in all directions — and can be approximated by spheres or ellipsoids — and “microcracks” — which are typically $10^2$ to $10^4$ times longer than wide. In rocks, for instance, pores can be due to intergranular spaces in sediments or to fluid inclusions in igneous rocks while microcracks can be cleavages through mineral grains or lie inside grain boundaries.

![Figure 2.4: Young's experiment for a homogeneous medium, a medium with an ellipsoidal pore and a medium with a penny-shaped microcrack. From [5]](image)

In mathematical terms, a crack is a boundary condition for the elastic strain field, with the additional complication that the precise shape of the crack is in principle unknown, and the shape depends on the stress field acting on it. To overcome this problem, one can prescribe
the crack geometry and then compute the stress field. This approximation can provide a useful guidance on the mechanism of crack propagation.

To understand how a microcavity modifies the mechanical behaviour of a solid, we perform Young’s experiment for a sample with one cavity (figures 2.4b and 2.4c) and compare it to a sample with no cavity (figure 2.4a). In the homogeneous sample of figure 2.4a every volume element contributes equally to the elastic response since on any cubic volume element $\Delta V$ three pairs of forces act in opposite directions having for each volume element the same absolute value $F$. If $\Delta S$ is the surface of one face of $\Delta V$ and $\sigma = F/\Delta S$ is the stress one can observe that in homogeneous systems the stress is everywhere the same.

Let us drill now a cavity into the medium. Those volume elements that would have been inside the cavity cannot contribute to the elastic response and the remaining volume elements must take over the excess load, i.e. the stress that would have acted on the missing volume elements. It happens, however, that the excess load is not shared equally by the remaining volume elements but that the redistribution of stress is strongly inhomogeneous. At the height of the cavity ($h$ in figure 2.4) the cross section is reduced and the stress acting on volume elements here is enhanced. But the increase of stress is not the same for all volume elements at the height $h$. If $\sigma_0$ is the stress acting on the volume elements before the cavity was drilled then the enhanced stress acting on volume element $k$ is $\sigma_k = K_T \sigma_0$ and we call $K_T$ the stress concentration factor.

For an elliptical hole in an infinite, linearly elastic medium (figure 2.4b) with major and minor radii $a, b$, the major being perpendicular to the load it can be shown analytically that at point $A$ in figure 2.4b

$$K_T = \frac{\sigma_k}{\sigma_0} = 1 + \frac{2a}{b} = 1 + 2\sqrt{\frac{a}{\rho}} \quad (2.31)$$

where $\rho = b^2/a$ is the radius of curvature at that point.

For two-dimensional plates analytical calculations can be pushed much further. So it is known that for an elliptical hole in an infinite, linearly elastic medium at height $h$ the excess stress, $(\sigma_k - \sigma_0)$, decays like $1/\sqrt{r}$ close to point $A$ while far from the hole the decay is that of a dipole field, namely $1/r^2$, being $r$ is the distance from point $A$. Close to $A$ one can even calculate for each of the three modes of figure 2.3 the full dependence on the angle $\theta$ as shown in figure 2.4b for each of the components of the stress.

It was first shown by Irwin [20] that close to the tip of a microcrack like the one shown in figure 2.4a the excess load also decays in a plate like $1/\sqrt{r}$. The proportionality constant $K$, defined through

$$\sigma_{ij} = \frac{K_T}{\sqrt{2\pi r}} f_{ij}(\theta, \phi) \quad (2.32)$$

where $K_T$ is the stress intensity factor and $f_{ij}$ is a crack-tip function in terms of the angular variables $\theta$ and $\phi$. It depends on the length of the crack, the geometry of the sample, and the
loading conditions. The value of \( K \) at which the crack breaks at the tip is the material-dependent constant, called "toughness" \( K_c \). For a given crack tip one has three toughness values, one for each of the three modes of figure 2.3. We see from the enhancement factor that the sharper the tip of a microcrack the more its stress is enhanced. At an infinitely sharp tip the stress has a singularity. So, even if the externally applied stress \( \sigma_0 \) is much below the breaking stress \( \sigma_c \), close to the tips of microcracks, the stress can be much higher than \( \sigma_c \) and the material can break at these tips. Once a microcrack tears open at its tip it becomes longer and therefore the stress is enhanced even more and it will not stop growing anymore.

The presence of singularity around the crack-tip, however, is a mathematical artifact of linear elasticity. On one hand the continuum description is only valid up to a lower cutoff at which the atomistic nature becomes visible. Consequently one cannot get arbitrarily close to the crack tip neither can a tip become arbitrarily sharp. On the other hand, real materials can not sustain arbitrarily large stresses and in general close to the crack-tip deformation is ruled by non-linear and inelastic effects. In practice we define a fracture process zone (FPZ) as the region surrounding the crack where these effects take place, while far enough from the crack one recovers linear elastic behavior, that is the long-range decay of perturbed displacement fields. One therefore has the interesting scenario that although the sample as a whole is elastic and breaks brittle there are regions inside the material, namely around the tips of microcracks, in which the material deforms plastically and breaks ductile. The shape and the mechanical response of the plastic regions around crack tips surrounded by an elastic medium is difficult to estimate. However, only the characteristic size \( \xi \) of these regions is essential for fracture because at \( \xi \), which is the shortest distance from the tip of the crack to the boundary of the plastic zone, the singularity of the elastic stress field is cut off. The largest value \( \sigma_c \) of the resulting stress field which is necessarily on this boundary has to be compared to the toughness of the material if one wants to know when the crack will grow.

### 2.6 Interacting cracks

We have discussed the behaviour of one single crack. Actually in the real world one gets the impression that most samples seem to be broken by just one crack. On the other hand, in most materials no particular crack can be singled out to be the critical crack that will break the system and on which the toughness and other quantities could be measured locally. In fact we started with a material full of pores or microcracks and therefore the next point to understand fracture is see how, among many equivalent randomly distributed microcavities, one crack will be born that will tear the system apart.

We consider ensemble of microcavities of arbitrary shape and orientation scattered in space as shown in figure 2.3. As discussed above each cavity produces a stress enhancement field
around it which depends on its geometry and size and that decays at large distances like a dipole field. Because of this slow decay (slower than exponential) one obtains a long range interaction. As long as one only considers linear elasticity the stress enhancement fields coming from the various microcavities can be added (superposition principle) and one obtains an extremely complex stress enhancement field for the ensemble which cannot always be handled, even numerically.

Phenomenologically the interplay of several stress enhancement fields can yield to the most diverse effects: between two close cracks aligned and perpendicular to the load like the pair shown in figure 2.5 at A the stress field is particularly enhanced. For long, straight cracks one can calculate that at the tip of one single crack of length $a$ the enhancement is proportional to $\sqrt{a}$ while between two cracks of length $a$ the enhancement is proportional to $a$. Consequently the two cracks shown at A in figure 2.5 will attract each other and if the external load is large enough coalesce, i.e. the neck between the two will break and they will become one single crack.

On the other hand, in other regions the stress field is substantially weakened like in the situation shown in figure 2.5 at B. There two cracks are parallel to each other and if the cracks are sufficiently long the stress between the two cracks is screened and the external load will not be felt. From screening to attraction all scenarios are possible specially if more than two cracks are present.

The interesting question is, however, not just the stress distribution generated by the ensemble of microcracks but the growth and coalescence of these microcracks when the external load is increased, i.e. its dynamical evolution. If we increase the load from zero and reach the point when the most stressed region fails (for instance that two close crack tips coalesce), due to this
coalescence the configuration of microcracks changes and this means that the stress enhancement field is modified. Although we only consider linear elasticity, the response of the system to the external load is now non-linear due to the changes in stress enhancement each time a region fails. Non-linear problems of this kind are very difficult to handle and practically all that can be done are numerical simulations.

While a precise calculation of the dynamical evolution of the ensemble of microcracks is technically difficult it is easy to give a qualitative description: the first regions that will fail, the most stressed one, are determined by the initial position of the microcracks, i.e. their random location. But once in a region a crack has become larger and therefore the stress has been locally enhanced this region is more susceptible for further cracking. So regions that failed once are prone to fail again. The effect of repeated failure compete with the failure of the randomly placed, stressed regions that did not yet fail. When the repeated failure takes over one is roughly at the critical breaking load. The growing cracks also compete against each other because the larger cracks grow faster. Coalescence of cracks is the fastest way of propagation. Finally one crack wins and spans through the system coalescing with the cracks it encounters on its way. The time it takes for one crack to impose itself over the others strongly varies with the configurations of microcracks and this explains the large sample to sample variations encountered in the postfailure regime.

2.7 Griffith's criterion

The description of brittle fracture given so far is not easy to use for quantitative predictions. Alternative, more phenomenological approaches to fracture have been developed and used over the years. The most popular one concentrates on energy conservation involved in fracture and was first formulated in 1921 by Griffith [1]. When a load is applied on a sample a potential elastic energy is stored in the system. If a new crack is formed or an existing crack grows part of this elastic energy is released. On the other hand the formation of a crack implies the creation of free surfaces. So Griffith's criterion states that a crack grows if and only if the release of potential energy is equal or larger than the surface energy that is required for the crack to grow. Let us show some more quantitative results.

Neither the elastic nor the surface energy are easily accessible in experiments or easy to calculate, but there are cases in which calculations are feasible like the previous case of one single elliptical flaw of length \(a\) perpendicular to an uniaxial external displacement that is imposed on a plate (see figure 2.7b). For mode I rupture one finds that, in the limit of large plates, the elastic energy change associated with a crack of length \(a\) is given by

\[
E_{\text{el}} = -\frac{\pi \sigma^2 a^2}{2E}, \tag{2.33}
\]
where $\sigma$ is the applied stress and $E$ is Young's modulus. The release of elastic energy caused by an elongation of the flaw $da$ is given by the energy release rate $G$:

$$G \equiv \frac{dE_{el}}{da} = \frac{\pi \alpha \sigma^2}{E}. \quad (2.34)$$

According to the Griffith’s criterion the crack will grow if $G$ equals the surface energy needed to create two surfaces of length $da$ (a crack in a plate has two sides), a process that has an energy cost proportional to the crack surface:

$$E_{surf} = 2ay_s, \quad (2.35)$$

where $\gamma_s$ is the specific surface energy. The crack will grow when the process leads to a decrease of the total energy $E = E_{el} + E_{surf}$, i.e. when $G$ exceeds the critical energy release rate $G_c$:

$$G > G_c \equiv \frac{dE_{surf}}{da} = 2\gamma_s, \quad (2.36)$$

The criterion for crack-growth can be also expressed as

$$-\frac{\pi \sigma^2 a}{E} + 2\gamma < 0, \quad (2.37)$$

which implies that under a stress $\sigma$ cracks of size

$$a > a_c \equiv \frac{\gamma E}{\pi \sigma^2} \quad (2.38)$$

are unstable. Equation (2.37) can also be written as $\sigma \sqrt{\pi a} > \sqrt{2E\gamma}$ and, noting that $K = \sigma \sqrt{\pi a}$ is the stress intensity factor of an infinite plate, we can reformulate the crack-growth criterion in terms of a critical stress intensity factor $K_c$, also known as the material toughness: the crack grows when

$$K > K_c \equiv \sqrt{2E\gamma}. \quad (2.39)$$

Knowing the surface energy which is a material constant one knows therefore the critical stress for mode I cracking. The critical energy release rate $G_c$ consequently plays a similar role as the toughness, i.e. depending on the rupture mode and on the material it determines when a single crack grows.

In simple geometries, it is possible to calculate the release of elastic energy of the medium due to the propagation of the crack and to relate this result to the stress intensity factor. This computation gives the equivalence between the approach developped by Irwin \[26\] and the previous one due to Griffith \[3\]. Let us mention the result for plane stress:

$$G_c = \frac{1}{E} \left[ K_I^2 + K_{II}^2 + (1 - \nu) K_{III}^2 \right]. \quad (2.40)$$

We have then defined two equivalent quantities, the energy release rate $G$ and the stress intensity factor $K$. 

2.7. Griffith’s Criterion
2.7.1 Limits of the Griffith’s criterion

Up to this point it seems that we have just described in terms of energies what we did before in terms of forces. Energies are, however, global quantities while forces are locally measurable and instead of a local equilibrium of forces we used the principle of energy conservation. Conceptually one is tempted to generalize the Griffith criterion to arbitrary systems with many cracks and to use $G_c$ as a global, phenomenological parameter. A drawback of this approach is that due to various effects happening in real materials energy conservation is not valid unless other contributions are taken into account. The viscoelasticity comes from energy dissipation within the material. Such a dissipation can be seen as an internal friction which can for instance heat up the system. In the energy balance dissipation makes up for an additional, unknown term that must be treated as another phenomenological constant and must be inserted by hand.

A particularly common loss of energy comes from the plastic regions around the crack tips since every plastic deformation requires work. A phenomenological term, the “plastic deformation energy” $\gamma_p$ can be introduced \[23,24\] and the Griffith criterion for fracture modified to $G_c = \gamma_s + \gamma_p$. Unfortunately in many cases this is not just a small correction because for most metals and polymers $\gamma_p$ is several orders of magnitude larger than $\gamma_s$. It seems nearly impossible to calculate $\gamma_p$ or $\gamma_s$ from first principles.

Another interesting problem related to the non-local nature of energy conservation is energy transport. The elastic energy freed by the growth of the crack is stored within a certain region and must be transported to the place where it is needed. There are several possible mechanisms that can assure the transport of elastic energy one being phonons, but the most common being Rayleigh waves on the crack surface. Each mechanism occurs with a certain velocity which for phonons is for instance the sound velocity, $(1.5 - 12) \times 10^3 \text{m/s}$ and which for Rayleigh waves is about $(1.5 - 12) \times 10^3 \text{m/s}$. The existence of a finite speed for energy propagation introduces a time delay into the energy balance which is important if very fast processes are considered. Into the Griffith criterion these effects introduce an additional kinetic energy term.

If $G > G_c$, i.e. if more energy is available than the minimum needed for crack growth, the excess energy is often transformed into kinetic energy for the propagating crack. Consequently the faster the crack grows the more excess energy is available. The existence of a finite velocity $v$ for energy transport imposes, however, a certain limitation on this picture. If the crack propagates faster than $v$ the elastic energy which is released behind the crack tip and which can therefore never reach the tip accumulates on the sides of the crack. This accumulated energy tends to create more crack surface leading to side branching of the crack. These side branches are usually short and thus called “microbranching”. If the velocity of the crack tip exceeds $\sqrt{2}v$ the crack tip can spontaneously bifurcate \[22\] and two cracks simultaneously propagating with speed $v$ can be sustained as long as both are equally fast and one does not outrun the other (“macrobranching”). Finally, if enough energy is available each tip can again bifurcate and so
one can finally have many simultaneously propagating tips.

2.8 Disorder

The influence of randomness on fracture can have many disguises. Leaving aside inelastic effects, we consider only, for the moment, brittle or “quasi-brittle” processes. In this case, we can generalize the discussion of the previous sections, using the concepts of long-range elastic fields, critical energy and crack size. There are several independent subcases that present different physics: one may consider a single crack and analyze the effect of disorder on material strength or, alternatively, one could study the dynamics of a whole crack population. In general, we can identify some issues to address:

i What is the effect of disorder on the stability of a single crack, and how does the Griffith argument get modified?

ii In the presence of many cracks, how is the physics changed from the case of a single crack?

iii What is the effect of disorder on the propagation of a single crack?

iv What are the concepts, and statistical laws that operate in the case of many cracks of which no single one dominates over the others?

These four fundamental questions also touch upon a number of engineering mechanics and materials science issues, such as fracture toughness, crack arrest, defect populations, damage mechanics, and failure prediction.

The first two issues from the above list deal with size-scaling and the statistics of strength. Figure 2.6 illustrates the schematic advancement of a two-dimensional crack. The presence of variation in the material structure can be presented ideally by locally varying elastic moduli, so that the stress-field becomes more complicated than the homogeneous LEFM solution. It can also be seen as a crack surface energy \( \gamma_s(x) \) which has a fluctuating component. The issue of size-effects amounts to compute the strength of a sample of linear size \( L \). The answer of course depends on whether the sample has one dominating crack or a larger number of microcracks or flaws. In the case of a single dominating defect, the Griffith argument can be easily modified to account for a spatial variation of the elastic coefficients. The technical problem is to consider the distributions of the surface energy \( \gamma_s \) and elastic modulus \( E \), or, even the presence of dislocations in the sample. As a result, one gets instead of the deterministic critical crack length (for constant \( E \) and \( \gamma_s \)) a probability for the crack to be stable under a given external stress. In other words, we can characterize the problem by a strength distribution which in the case of a single crack crucially depends on the microscopic randomness. For instance, Arndt and Nattermann [23]
showed that an uncorrelated random $\gamma_s$ results in an exponential strength distribution. In general, however, $\gamma_s$ could also display spatial correlations, as measured by the two point correlator $\langle \gamma_s(x)\gamma_s(y) \rangle$. Correlations in $\gamma_s$ imply that an increased crack opening will cost energy that not only depends on $\gamma_s$ at a single location but also depends on the presence of such correlations.

### 2.9 Damage mechanics

Some materials under stress are subject to a decrease of their elastic moduli with or without showing signs of plastic deformations. This decrease of the elastic moduli is due to the creation of microcracks in the material, and is generally called damage.

The discourse on strength changes if one allows for slow average crack growth. Consider the case in which the fracture dynamics is continuous and slow, so that a sample can be considered in a quasi-equilibrium. This means that the sample internal state is either independent of time or evolves slowly with respect to the time-scale of the fracture process. Even in these conditions, treating the dynamics of interacting microcracks explicitly represents a formidable task: the growth of a microcrack can be either inhibited/screened or accelerated due to the presence of a neighboring microcrack. The outcome, in general, depends on the loading conditions, the crack geometry, and other details as discussed in the engineering mechanics literature.

To avoid all these complications, one can disregard the dynamics of individual microcracks and treat the damage at a coarse-grained scale. This approach, known as damage mechanics [18,13,24], is valid when the interaction among cracks is moderate. To describe the mechani-
ical behaviour, we need to introduce an internal variable which characterizes the irreversible evolution. Let us call $E_0$ the initial Young modulus of the material, with no damage. When some microcracks have been created in the material, the apparent Young modulus is $E = (1 - D)E_0$. The damage $D$ is generally a fourth-order tensor, but in the simplest case, one can consider the evolution of a scalar-valued damage field $D(x)$ which goes from one to zero describing the local variations of the Hooke’s law

$$\sigma(x) = E_0(1 - D(x))\epsilon. \tag{2.41}$$

The idea behind damage mechanics is that all the complications of the fracture process can be encoded in the field $D$, that will then evolve according to a prescribed law, reflecting the internal microcracking in the material.

Simple models of statistical fracture can be used to check some of the typical assumptions and possible outcomes in damage mechanics. This is in particular applicable in scalar fracture, described by the electrical analogue of the RFM. In order to apply the coarse-graining procedure, implied by equation (2.41), one assumes the presence of a length-scale over which the damage self-averages, meaning that its fluctuations become much smaller than the average. This scale is also known as the representative volume element (RVE) above which it is possible to treat the damage as a smoothly varying variable.

Figure 4.2 depicts the typical scenario for damage accumulation. In this example we imagine a set of microcracks that will grow in a stable manner, and give rise to locally enhanced damage $D(x)$. One may now naturally define a correlation length, by measuring e.g. the two-point correlation function of a scalar $D$ via

$$C_D(r) = \left\langle \left( D(x + r) - \langle D \rangle \right) \left( D(x) - \langle D \rangle \right) \right\rangle^2 \tag{2.42}$$

which should decay, for a field $D$ which is uncorrelated over longer distances, asymptotically exponentially and thus defines a correlation length $\xi$ via $C_D \sim e^{-r/\xi}$. An interesting issue is the which should decay, for a field $D$ which is uncorrelated over longer distances, asymptotically exponentially and thus defines two correlation length $\xi_\parallel$ and $\xi_\perp$.

Given a fracture process, in which damage grows in a controlled-fashion, one can then, in principle, follow the development of $\xi$. This quantity will increase, making a sample effectively smaller: as $\xi$ (and the RVE size) increases $L/\xi$ diminishes, where $L$ is as usual the sample linear size. An additional complication comes from the damage anisotropy, since the two correlation lengths $\xi_\perp$ and $\xi_\parallel$ will follow different dynamics.

It is interesting to discuss what happens when one of the two becomes much larger than the other. In this case the sample becomes effectively much larger along the stress direction, becoming almost one-dimensional, which in other words indicates the presence of damage localization. Consider for instance the mechanics of fiber-reinforced composites: eventually one crack will dominate, but still the specimen may have a finite strength remaining thanks to fibers that bridge the crack and induce cohesion.
### Three

#### Fracture modeling in disordered media

3.1 The importance of disorder

Disorder comes into play in many ways during a fracture process. Let us just mention again that fracture usually enhances an initially present disorder through the nucleation of new cracks, or simply due to the heterogeneity of the stress field that results from the complex topology of the existing cracks. Small initial disorder can be enormously amplified during fracture. There are some basic experimental facts that will show that disorder is an unavoidable concept if one wants to deal with real materials, particularly brittle ones.

- Statistical fluctuations in the rupture stress of materials are very often enormous. Two pieces made of the same material, under the same conditions, will not break at the same time. These sample to sample fluctuations are present in most materials, and at all scales.

- Besides the statistical fluctuations, the mean fracture stress depends on the sample size. In general it decreases. Similarly, if the system size remains constant, but if the characteristic size of the microstructure changes, the mean fracture stress will be affected. These size effects tell us that we cannot neglect the small scale structure of the medium.

From these experimental facts, we know that it is not possible to reduce the behaviour of a material to its average. Fluctuations are fundamental, not only because of the necessary statistical analysis of failure in many practical instances, but also because these fluctuations at a local scale provide a necessary stabilizing mechanism to prevent the localization of damage. How to take into account these effects? There are two main routes: we want to describe the mechanical behaviour, or we want to understand it. Obviously, we would appreciate to do both at the same time. But the tools that are best suited to one of these approaches will be either inefficient or very heavy for the other one. Let us emphasize the complementarity of both approaches. Taking disorder into account explicitly means that we have to do a lot of statistics, or to deal with large system sizes to describe accurately a situation. This is computationally very expensive, and cannot be considered as a general way to handle engineering problems. However, this approach
can yield much information on size effects and the way they should be modelled using a damage theory. The length scale appearing in damage theory with localization limiters may eventually be obtained through these methods. The general statistical distribution of failure probability can also be obtained in this way. Therefore, even if the philosophy of the approaches is different, the results obtained should be complementary.

3.2 Approaches to fracture modeling

How can we try to obtain some information at the macroscopic level starting from a microscopic one and taking into account local fluctuations due to disorder? In elasticity of heterogeneous structures, homogenization of periodic structures has proved to be very useful. However, no disorder is present in the description, and since periodicity is a fundamental ingredient of this technique, it seems hopeless to extract any relevant information concerning heterogeneities using this tool. Among the various ways of handling disorder directly, we can, in general, distinguish three different approaches: mean field, local models, and renormalization.

Mean Field: the mean-field approach consists in neglecting the detailed spatial arrangement around a given volume element. The surrounding of a volume element is represented by a homogeneous embedding medium. In this way, each element interacts equally with all others, and these interactions are represented by the fictitious “equivalent” homogeneous medium. These conditions imply that mean field will be a good approximation

1. when interactions are long-ranged;

2. for problems which are not strongly sensitive to the presence of disorder;

3. at high dimensions.

Fracture is much more sensitive to disorder than elasticity and therefore mean field is expected to give poor results in usual space dimensions. In addition, fracture naturally enhances the effect of the preexisting heterogeneities in such a way that the initial disorder cannot be neglected. The final rupture results from the complex interplay between the initial disorder and the heterogeneities created by the fracture itself. Some models tend to indicate that the final stage of rupture can be interpreted as a critical point \[1\]. This fact raises serious doubts about conclusions reached about the final stage of rupture from a mean-field approach.

Local: a very different spirit stands behind the local approach where one tries to reproduce as closely as possible the real microscopic behaviour starting from first principles. This is for instance the case for molecular dynamics. Once the interaction between atoms has been chosen,
the general molecular dynamic framework should provide naturally elasticity, plasticity, viscosity, damage, and fracture properties without having to insert into the model any rheological behaviour by hand. This approach applied to fracture is in general so heavy that only results on small system sizes can be obtained, and therefore no firm conclusions can be drawn from these studies.

**Renormalization:** statistical physics has introduced a very powerful tool to study critical phenomena which exploits self-similarity: renormalization. It consists in expressing the invariance of some physical quantities under a change of length scale. In the framework of field theory, this method has provided many results for critical phenomena. Although this framework seems rather well fitted to our problem, attempts in this direction are very difficult in realistic cases.

This seems a pessimistic picture, but many approaches follow an intermediate route. The spirit of these studies is to describe very simply the mechanical behaviour of microscopic — or mesoscopic — elements, and to consider the collective behaviour of these elements in the presence of disorder (i.e. random variations of the local properties from one element to the next, or randomness in the initial state or during the time evolution, etc.). This approach is "local" in the previous sense but it contains some strong simplifications of the elementary behaviour.

### 3.3 Molecular dynamics

The most detailed and realistic simulation of fracture is certainly the atomistic one. In the ideal case one knows the electronic structure of the atoms and the crystal structure and can deduce from there the interatomic forces. Using Newton's law one can then calculate numerically the motion of each atom and watch how the crystal deforms under an externally applied force. In this way one could study the formation and propagation of a crack on a very microscopic level. Unfortunately this problem is too ambitious in view of the lack of information about the electronic structure and most importantly considering the existing limitations of computational capabilities. Only recently there have been attempts to correlate the results of electronic structure calculations with the mechanical response [25] and one is still far from having a first principles derivation of the interatomic forces that hold the crystal together. For this reason one is forced to use heuristic interatomic potentials which in the best of the cases can be matched to experimental neutron diffraction dispersion data.

The first applications of molecular dynamics to fracture were done in the early seventies to study the forces needed to open a single straight crack in a lattice. Starting with one-dimensional chains modeling the crack surface [26] there were soon results available obtained for diamond lattices and using rather complicated, angle-dependent potentials [27]. In this way predictions can be made for monocrystals of very specific materials. Instead of trying to model in more and more detail specific substances it is also very interesting to understand generic features of
fracture that should be applicable to a wide class of materials although such an approach will not yield to precise quantitative predictions for given experimental situations. In this spirit a good and very commonly used interatomic potential is the Lennard-Jones (6-12) potential

$$\phi_{ij}(r_{ij}) = \epsilon \left[ \left( \frac{d}{r_{ij}} \right)^{12} - 2 \left( \frac{d}{r_{ij}} \right)^6 \right]$$  \hspace{1cm} (3.1)

where all energies are measured in units of $\epsilon$ and $d$ is the equilibrium distance between particles. For computational reasons this potential is usually cut off at some distance that could be $1.6d$ which includes next-nearest neighbours and which is well beyond the distance $1.11d$ where the interatomic force is maximum.

### 3.4 Lattice Models

A lattice formulation of fracture, alternative to molecular dynamics, is given by discretizations of continuum equations. In this case the medium is reduced to a set of points embedded into a grid. Only local laws, like the balance of force and momentum, are considered and their implementation involves for each point only a few neighbours. Mathematically the calculation of collective properties then reduced to solving a set of coupled linear equations. Evidently these methods do not pretend to describe nature on an atomistic level as molecular dynamics but their validity is at much larger length scales where the medium can be described by continuous vector fields. The breaking of the lattice is not a natural consequence of the simulation as it is the case in molecular dynamics but has to be put into the model by hand as an additional rule of the model. This rule can lean on experimental data or on phenomenological laws.

One big advantage of the lattice models is that they allow very naturally for the introduction of disorder. An even bigger advantage is that they are more “rigid” than a crystal in molecular dynamics and the breaking of a bond is an irreversible yes-no decision. Consequently already small cracks are sharply defined and it is possible to simulate simultaneously many cracks within rather moderate lattice sizes. Lattice models are therefore good candidates to overcome the numerical difficulties that we encountered in molecular dynamics.

#### 3.4.1 General structure of lattice models

All the models that we consider here have some common setting which can be summarized in the following way: the medium is discretized such that all (spatial) sites are equivalent. Each site has the same number of neighbours (coordination number $z$) so that one has the topological structure of a regular lattice. The variables that characterize the medium (the electric field for dielectric breakdown, the displacement vector for elasticity, etc.) are placed on the sites of the lattice. The equations that describe the medium (Laplace for electric models (2.28), Lamé for elastic models (2.16), etc.) are discretized so that for each site one has one equation per variable.
which only involves variables on the \( z \) neighbour sites. The continuous equations of the medium are therefore transformed into a set of \( vN \) coupled linear equations, where \( v \) is the number of variables per site and \( N \) the number of sites of the lattice. Since only nearest neighbours are involved in each of the equations a solution of the set of equations only involves the inversion of a sparse matrix. Only the boundary condition of the outer boundary on which the externally imposed constraint is applied is explicitly implemented. After implementing the boundary conditions the set of equations has a unique solution.

The simulation of a rupture process must be done in an iterative way: the equations must be solved to determine which bond (or bonds) should be broken, but once the bonds is broken the solution of the equations is changed. Consequently the equations must be solved again if one wants to know which bond to break next and so on.

The algorithm performed in one iteration can be decomposed into five steps:

1. solving the set of equations;
2. determining the set of all the bonds that are eligible to be broken;
3. calculating for each bond of this set a certain quantity \( p \) which is a function of the solution of the equation on the sites adjacent to the bond;
4. choosing, according to a rule that depends on \( p \), which bond will be "broken";
5. breaking the bond, changing its conductivity or elastic modulus in the equations.

Each of these steps allows for a large variety of options that contain many possible physical situations. 1) describes the nature of the medium and the externally applied constraints, 2) the connectivity of the crack, 3) and 4) the breaking rule and the disorder and 5) distinguishes for instance between a breakdown and a fuse problem.

### 3.4.2 Boundary conditions

Two types of models have been studied: scalar ones and vectorial ones. In all cases one studies an equation which in its most general form is \( \nabla (\epsilon \nabla \phi) = 0 \). In the electric case \( \phi \) is the electric potential and \( \epsilon \) the material dependent dielectric constant (or tensor). In principle, \( \epsilon \) can vary from site to site modelizing spatial inhomogeneities of the material. In most cases \( \epsilon \) is considered to be a constant so that the equation simplifies to \( \Delta \phi = 0 \).

Two types of external boundary conditions are usually used in scalar media, either a circular geometry (figure 5.1a) or a uniaxial one (figure 5.1b). The first case is typically used for dielectric breakdown \([28]\) while the second case has been applied to random fuse networks \([7]\).

Vectorial models have been considered up to now only for elasticity. So, only equations for the displacement field have been investigated. However, as opposed to the scalar case there are
at least two possible equations for elasticity, namely the Lamé and the Cosserat equations. In addition, various rather different discretizations for the equations have been in use: the central-force model \[29\], the bond-bending model \[30\] and the beam model \[31\].

Vectorial models allow for a larger variety of externally imposed constraints than scalar models. At least four different types of external boundary conditions have been considered: Shear \[32\] (figure 3.3a), uniaxial tension or compression (figure 3.2b), uniform dilation \[32\] (figure 3.2c) and surface cracking \[33\].
### 3.5 Breaking rule and disorder

Most of the microscopic physics of rupture is contained in the breaking rule. As observed in the previous section, the breaking rule has to nature of lattice models as opposed to molecular dynamics. One has freedom in the choice of the breaking rule and its parameters allows to carefully explore the impact of a given microscopic mechanism on the macroscopic behaviour of the system.

The effect of disorder on the breaking process is very important and for this reason its implementation within the breaking rule should be discussed in some detail: real materials exhibit the most diverse types of disorder, ranging from small deviations of the crystalline order, like vacancies, to the large scale heterogeneities of composites. This disorder can also be in motion: dislocations can migrate, microcracks can form and heal, interstitials can diffuse, etc. The fundamental distinction that has to be made for our theoretical purposes is, if this motion is much slower than the investigated process (i.e. fracture) or not. In the first case the disorder is called “quenched” and can be considered time-independent. In the second case, called “annealed”, the interplay of disorder fluctuations and fracture has to be taken into account.

On the mesoscopic length scales that lattice models are supposed to describe one can essentially reduce the description of disorder to a spatial dependence of the local density, elastic modulus or strength. In the case of quenched disorder this spatial dependence is fixed once and forever before the rupture process starts. Since we ignore the microscopic details and since we adhere to simplicity the spatial dependence is chosen to be random according to some probability distribution. This distribution contains, among others, information about the degree of disorder so that by considering various distributions the effects of disorder on rupture can be quantified.

In the context of lattice models the individual bond represents the system on a mesoscopic scale and is therefore characterized by a constitutive law which in the simplest case should just be linear up to a breaking threshold value. The strength of the bond is given by the threshold value and its conductivity or elastic modulus by its slope. The presence of disorder can be implemented by allowing the bonds of the same lattice to have different constitutive laws. Strength fluctuations can be represented by choosing the threshold value for each bond randomly \([5, 15]\). Spatial variations in the conductivity or elastic moduli can be described by randomly choosing for each bond different slopes \([15]\).

The most prominent case of spatial density fluctuations are porous media where a given point in space is either massive or empty. This situation can be modelled by randomly taking out a bond of the lattice with a probability \(q = 1 - p\) before the breaking process starts and has been extensively studied for electrical \([2, 17, 57]\) and mechanical \([58]\) networks. Particularly interesting is the behaviour of these “dilute” systems close to the percolation concentration \(p_c\).

In quenched systems the random variables are fixed at the beginning and the following rup-
ture process is completely deterministic: to break a bond the equations are solved and the bond for which the ratio strain over threshold \( \delta_c \) is largest is chosen to be broken.

### 3.5.1 Other features of breaking rules

As opposed to electric fuses the elastic bonds, represented by beams or springs, can break in various modes. If the elastic bond is a rubber band it will tear apart when stretched but if it is a glass rod it will crack when bent. In three dimensions another mode of rupture appears, namely torsion. Mechanical breaking rules should reflect these various rupture modes. This can be achieved by defining a quantity \( p \) that determines the breaking as a sum of terms each corresponding to one mode. In the quenched case for instance one can consider that the beam breaks for which

\[
p = \left( \frac{f}{f_f} \right)^2 + \frac{\max(|m_1|, |m_2|)}{I_m}
\]

is largest [39].

Damage can also be modelled within the breaking rule by lowering the breaking threshold of bonds in the vicinity of the crack by an amount proportional to the strain without breaking them. Bonds damaged in this way are more likely to break at one of the next iteration steps [40].

### 3.5.2 Bond breaking

Once the bond to be broken has been chosen its actual removal or replacement has to be implemented. In the case of fracture or fuse networks one can simply take the bond out, i.e. remove the terms corresponding to this bond from the set of equations describing the medium. This is equivalent to setting the elastic modulus or the conductivity of the bond to zero.

It is, however, also possible to consider effects of residual strength by just reducing the elastic modulus or conductivity of the bond by a certain amount. In this case the interesting phenomenon of crack arrest has been observed [41].

### 3.6 Fiber bundle models

One of the simplest approaches to analyze fracture in disordered media is represented by the study of fiber bundle models [8,9]. These models were originally introduced to describe fibrous materials, schematizing the sample as a set of brittle fibers loaded in parallel. As in other lattice models, the failure threshold of each fiber is randomly chosen from a distribution. Next, one has to impose a rule for load redistribution after each failure. The simplest possibility is the case of an equal load sharing (E.L.S.), in which each intact fiber carries the same fraction of the load. This case represents a sort of mean-field approximation and allows for a complete analytic
3.6. Fiber Bundle Models

At the other extreme lies the local load sharing (LLS) model where the load of a failed fiber is redistributed to the intact neighboring fibers [46-48].

3.6.1 Equal load sharing

We consider first the case of ELS fiber bundles, in which $N$ fibers of unitary Young’s modulus $E = 1$ are subject to an uniaxial load $F$. Each fiber $i$ obeys linear elastic equation up to a critical load $x_i$, which is randomly distributed according to a distribution $p(x)$. When the load on a fiber exceeds $x_i$, the fiber is removed. Due to the ELS rule, when $n$ fibers are present each of them carries a load $F_i = F/n$ and consequently a strain $\epsilon = F/n$. The constitutive law for ELS fiber bundles can be easily be obtained from a self-consistent argument. At a given load $F$, the number of intact fibers is given by

$$n = N \left[ 1 - \int_0^F p(x) dx \right].$$  \hspace{1cm} (3.3)

Rewriting equation (3.3) as a function of the strain, we obtain the constitutive law

$$f = \frac{F}{N} = \epsilon (1 - P(\epsilon)), \hspace{1cm} (3.4)$$

where $P(x)$ is the cumulative distribution obtained from $p(x)$.

As a simple illustration, we consider a uniform distribution in $[0, 1]$, so that equation (3.4) becomes

$$f = \epsilon (1 - \epsilon). \hspace{1cm} (3.5)$$

Similarly we can obtain the fraction of intact fibers $\rho = n/N$ from equation (3.3) $\rho = 1 - f/\rho$ which can be solved to yield

$$\rho = \frac{1 + \sqrt{1 - 4f}}{2}. \hspace{1cm} (3.6)$$

This equation shows that as the load is increased $\rho$ decreases up to $f_c = 1/4$ at which $\rho = \rho_c = 1/2$. For larger loads equation (3.6) displays no real solution, indicating the onset of catastrophic failure.

It is interesting to rewrite equation (3.6) as

$$\rho = \rho_c + A(f_c - f)^{1/2}, \hspace{1cm} (3.7)$$

with $\rho_c = 1/2$, $f_c = 1/4$, and $A = 1$.

In fact, this form is generically valid for most distributions $p(x)$. To see this, we rewrite equation (3.3) as $f = x(1 - P(x))$ where $x = f/\rho$ is the load per fiber. Failure corresponds to the maximum $x_c$ of the left-hand side, after that there is no solution for $x(f)$. Expanding close to the maximum we obtain $f \approx f_c + B(x - x_c)^2$, which then leads to equation (3.6) as long as the
distribution is sufficiently regular. This law also implies that the average rate of bond failures increases very rapidly before fracture:
\[
\frac{dp}{df} \sim (f_c - f)^{-1/2}.
\] (3.8)

The preceding discussion focused only on average quantities, but the presence of a random distribution of failure thresholds necessarily leads to fluctuations. Due to the ELS rule, however, strength fluctuations are not particularly interesting and vanish in the limit of large bundles. In particular, for any threshold distribution such that \(1 - p(x)\) goes to zero faster than \(1/x\) for \(x \to \infty\), the strength distribution is Gaussian with average \(f_c = x_c(1 - P(x_c))\) and standard deviation \(\sigma = x_c p(x_c)(1 - p(x_c))/\sqrt{N}\).

### 3.6.2 Local load sharing

The case of LLS represents another extreme case in which the effect of long-range stress field is completely neglected, but stress enhancement around the crack is treated in the simplest way. Consider for instance a 1D series of fibers loaded in parallel with random breaking threshold from a distribution \(p(x)\). When the load on a fiber exceeds the threshold its load is redistributed to the neighboring intact fibers. Thus the load on a fiber is given by \(f_i = f(1 + k/2)\), where \(k\) is the number of failed fibers that are nearest neighbors of the fiber \(i\) and \(f_1 = F/N\) is the external load [cite256]. Even for this apparently simple 1D model a closed form solution is not available, but several results are known from numerical simulations, exact enumeration methods or approximate analytical calculations. The analysis of the LLS model is extremely complicated and we thus list here the main results obtained, referring the reader to the relevant literature for the details \cite{55-56}.

Contrary to the ELS model, LLS fiber bundles normally exhibit non-trivial size-effects as could be anticipated from general consideration of extreme value statistics. In particular, the average bundle strength decreases as the bundle size grows as
\[
f_c \sim \frac{1}{\ln(N)},
\] (3.9)
so that an infinitely large bundle has zero strength.

In the limit of large \(N\), it has been shown that the strength distribution should follow the form
\[
W(f) = 1 - [1 - C(f)]^N,
\] (3.10)
where \(C(f)\) is a characteristic function, close to the Weibull form, but difficult to determine exactly \cite{48}. The existence of a limit distribution has been recently proved under very generic conditions for the disorder distribution in \cite{57}. This result implies that in LLS bundles there is no disorder-induced transition from brittle failure, ruled by extreme value statistics, to a more gradual tough regime where fibers break gradually.
3.6.3 Generalizations of fiber bundle models

Fiber bundle models with ELS and LLS represent two extreme idealizations of fibrous materials that provide some insight on the general failure properties of disordered systems. These models have been improved in various ways, to obtain a more realistic representation of fibrous composites through a more detailed description of their mechanics and by including the effects of disorder on more complicated constitutive laws. There is a vast literature on the effect of the fiber arrangement, for instance 2D arrays rather than a 1D chains have been considered. Furthermore, it is possible to interpolate between LLS and ELS behavior through a long-range load transfer rule [58, 59], or by a mixed mode in which a fraction of the load is transferred locally and the rest globally [60].

3.7 Network models

3.7.1 Random fuse networks: brittle and plastic

![Figure 3.3: RFM with diamondlike lattice topology. Each of the bonds in the network is a fuse with unit electrical conductance and a unit breaking threshold. A fraction of the fuses is randomly removed from the system. The behavior of the fuse is linear up to the breaking threshold. Periodic boundary conditions are applied in the horizontal direction and a unit voltage difference, \( V = 1 \), is applied between the top and bottom of lattice system bus bars. As the current \( I \) flowing through the lattice network is increased, the fuses will burn out one by one until the system reach the failure (showed on the right).]

The simplest truly interacting fracture models are variations of the scalar analogy of fracture captured by the RFM [7, 59]. The equations governing the RFM can be considered as a discretization of the continuum Laplace equation (2.28)

\[
\nabla^2 V = 0
\]

(3.11)

with appropriate boundary conditions (see figure 3.3). This scalar electrical analogy of elasticity is a simplification over the Lamé equations (equation (2.16)) described in §5.1, and formally
corresponds to an antiplanar shear deformation scenario. In the discrete version, the currents and voltages over the nodes $x_{ij}$ satisfy the Kirchhoff and Ohm’s laws and depend on the local conductivities $\sigma_{ij}$ for the links (or resistors between the nodes). The fuse is then simply an element with a property of irreversibility: once a critical current (or similarly a voltage drop across the fuse: if $\sigma_{ij} = 1$ the two cases are equivalent) $i_c$ is reached, the conductivity goes to zero. Clearly, a model with such fuses is appropriate to describe the failure of very brittle media. The failure becomes more gradual in the presence of disorder, since not all the links would fail at the same time. Although much of our focus is on the RFM, many of the issues that we consider are directly applicable to other discrete models as well.

Quenched randomness is usually introduced in the RFM in different ways:

1. **random thresholds**, extracting the thresholds $i_{c,ij}$ of each link from a probability distribution $p(i_c(x))$;

2. **random dilution**, removing a fraction $p$ of the links at the beginning of the simulation;

3. **random conductivities**, extracting the conductivities $\sigma_{ij}$ of each link from a probability distribution $p(\sigma(x))$.

All these three methods allow tuning the amount of disorder present in the system by changing the distributions or the dilution ratio.

The crossover from weak to strong disorder is mostly studied in the framework of the random thresholds (case 1) using either a uniform distribution in $[1 - r, 1 + R]$ or a power-law distribution $p(x) \sim x^{-1+1/\Delta}$ in $[0, 1]$. The first case interpolate between no disorder ($R = 0$) to strong disorder ($R = 1$), while the second case allows for extremely strong disorder. The effect of percolation type disorder (random dilution, case 2) on fracture will be explored in section §4.2. The scenario of quenched random conductivities poses an interesting question on how different disorder phases would need to be defined in the case of a RFM. In general, analogous to elastic media with locally varying elastic constants or simply percolative disorder, quenched conductivity variations create correlations among local currents $i$ even in the linear $I-V$ regime.

The brittle RFM scenario is usually considered in the limit of time-scale separation. This implies that the equilibration time of the currents is much faster than the ramp-up rate of external potential or current. The dynamics is then defined by finding the quantity

$$i_{ij}^* = \max_{ij} \frac{i_{s,ij}}{i_{c,ij}}$$

at each timestep $t$. This is an example of extremal dynamics. Here, the physics ensues from the combination of local current enhancements and disorder, in the thresholds $i_c$. 


Figure 3.4: The schematic of a $I-V$ curve for the RFM. Two alternatives, current- and voltage-driven, are outlined. The schematic corresponds to a failure process in which one bond at a time fails \[\text{[63]}.\]

Figure 3.4 shows a schematic of the $I-V$ behavior of an RFM. Note that the series of fuse failures defined by equation (3.12) is unique given the set of thresholds $i_i$. The $I-V$ characteristics depends, however, on the way the system is driven: two different $I-V$ curves ensue depending on whether the system is driven by constantly increasing current or voltage. The figure implies that in the voltage driven scenario, there is the possibility of a weakening tail (the regime of $I < I_{\text{max}}$ and $V > V(I_{\text{max}})$, whereas in the current-driven scenario, the curve stops at $(V,I_{\text{max}})$.

Hence, depending on the loading scenario, there are two failure points. Under current control all the fuses in the post-peak regime (beyond the peak current $I_{\text{max}}$) break at once, whereas in the voltage case, fuses in the post-peak regime fail gradually. The schematic presented in figure 3.4 corresponds to a typical simulation in which failure of one bond at a time occurs.

It is interesting to cite that there is a way to interpolate between fully brittle and plastic response. Instead of removing a fuse, it is possible to just reduce its conductivity by $\sigma_{ij} \rightarrow (1-D)\sigma_{ij}$, where $D$ is the damage. This leads naturally to crack arrest \[\text{[41]},\] and thus to distributed damage.

Finally we notice that there are some experimental studies of systems corresponding to realizations of the RFM. Otomar et al. \[\text{[64]},\] built a network of fuses similar to the numerical models RFM, while a thermal fuse model has been worked out experimentally in \[\text{[65]}.\]

3.7.2 Tensorial models

Following the general strategy of the RFM, several models with more realistic elastic interactions have been proposed in the past. The simplest possibility is provided by central force systems,
where nodes are connected by elastic springs \[66-69\]. The random spring model (RSM) is defined by the Hamiltonian

\[ H = \sum_{ij} \frac{K}{2} (u_i - u_j)^2, \] (3.13)

where \(u_i\) is the displacement of node \(i\) and \(K\) the spring constant. The elastic equilibrium is obtained by minimizing equation (3.13), and disorder is introduced in the standard way by either random dilution, random threshold, or random elasticity.

A similar but slightly more complicated model is the Born model, defined by the Hamiltonian

\[ H = \frac{1}{2} \sum_{(i,j)} \left[ \alpha (u_i - u_j)^2_{\|} + \beta (u_i - u_j)^2_{\perp} \right]. \] (3.14)

where \((i, j)\) represents the nearest neighbors of the node \(i\), and the summation is carried out over all the nodes in the lattice system. The quantities \((u_i - u_j)_{\|}\) and \((u_i - u_j)_{\perp}\) represent the relative displacement of the node \(j\) in the directions parallel and perpendicular to the bond \((i, j)\) respectively. In this case, there is a primitive competition between stretching and bending the bond between two nodes on a lattice. In the limit \(\alpha = \beta\) the Born model reduces to the random spring network (if \(u\) is scalar, it also reduces to the random fuse network), but otherwise it still lacks one fundamental property of real elastic systems that local rigid rotations cost energy (see [59, 70]).

The most used (and most physically correct) models in this respect are the so-called beam model and bond-bending models. One may view this kind of a system as a lattice of massless particles interacting (or connected) by a beam to their nearest neighbors \[61\]. In the beam models, each beam is capable of sustaining longitudinal \((F)\) and shear forces \((S)\), and a bending moment \((M)\).

The failure condition of these models depends on the combination of local torque, longitudinal, and shear forces in beam models, or on central and angular forces in the bond bending model. In the beam case a typical fracture criterion is of the type

\[ \left[ \frac{F}{t_F} \right]^2 + \frac{\max(|M_i|, |M_j|)}{t_M} \geq 1 \] (3.15)

where \(t_F\) and \(t_M\) are force and moment thresholds respectively, which can be chosen separately for each bond. Similar to the RSM, one can then tune the system by scaling the external load so that only one bond fails at each time. In addition, it is possible to control the relative roles of the two fracture mechanisms by either adjusting the magnitudes of \(t_F\) and \(t_M\) or adjusting the beam elasticity.
Four

Statistical theory for fracture models

4.1 Statistical mechanics of cracks

The idea that there is a relation between fracture and phase transitions has a long history. For instance, the Griffith theory of fracture is very similar in spirit to the classical theory of nucleation in first-order phase transitions. In the following we provide a basic introduction to the phase transitions and discuss its relation to fracture using lattice models as an illustration.

4.1.1 Generalities on phase transitions

Phase transitions are characterized by changes in the internal symmetries of a material as external control parameters are varied. Familiar examples are the melting of a crystal, or the ferromagnetic transition in a magnet. In the first example, we have an abrupt first-order phase transition, with latent heat, coexistence and no precursors, while the latter is a continuous second-order transition.

To be more quantitative, one typically distinguishes a phase by an order parameter whose value is related to the internal symmetries of the system. For instance, a ferromagnet acquires a non-zero magnetization when the spin rotational symmetry of the paramagnetic phase is broken. Since the transition is continuous, the magnetization vanishes as the transition is approached. This is unlike first-order transitions, where the order parameter is discontinuous across the transition.

We consider the example of a uniaxial ferromagnetic system. The control parameters, in this case, are the temperature $T$, the magnetic field $H$, and the average magnetization $M$, which is the order parameter of the system. At high temperatures, fluctuations dominate and the magnetization is zero on an average. As the temperature is lowered, the local magnetization becomes more and more correlated until the critical point $T = T_c$ is reached. The correlation length $\xi$ then diverges as

$$\xi \sim (T - T_c)^{-\nu}, \quad (4.1)$$
where \( \gamma \) is a critical exponent. The average susceptibility \( \chi \) also diverges, indicating that the system is extremely prone to changes, and it does so as

\[
\chi \sim (T - T_c)^{-\gamma}.
\]

The order parameter \( M \) becomes non-zero in the ferromagnetic phase and close to the critical point follows

\[
M \sim (T - T_c)^\beta,
\]

defining another critical exponent. The presence of a magnetic field destroys the transition, forcing a particular direction for the magnetization, and thus defines another scaling law at \( T = T_c \)

\[
\phi \sim H^{1/\delta}
\]

We have defined a certain number of critical exponents, which are however not all independent, as can be shown by a more detailed analysis. Here, we do not want to go further into these details, but just highlight the fact that second-order, phase transitions are associated with scaling and a diverging correlation length \( \xi \).

### 4.1.2 Phase transitions in fracture models

The interpretation of fracture as a phase transition is a highly controversial question. While at first glance the analogies are striking, a precise relation is hampered by several difficulties and no consensus has been reached in the literature on this subject. A series of studies consider the case of crack nucleation in homogeneous media, elaborating on the analogy between the classical theory of nucleation and Griffith theory.

Several authors have tried to relate the scaling laws observed in fracture morphologies and acoustic emissions (\( \Delta \)E) distributions to an underlying critical point, basing their analysis on experiments [71, 72] and models [43, 73-77]. A natural starting point is provided by a lattice model, which could be mapped into other models, typically drawn from ferromagnetism, where the presence of a phase transition is well established [73, 77]. It is instructive to consider first the fiber bundle model, since exact results are available and the nature of the hypothetical phase transition can be clearly identified [73, 74]. We can assign to each fiber a spin variable \( s_i \), whose value depends on whether the fiber is intact \( (s = -1) \) or broken \( (s = 1) \). The effective field is formally given by

\[
k_{i}^{\text{eff}} = F_i(s_j, F) - x_i \tag{4.5}
\]

where \( x_i \) is the fiber threshold, \( F_i \) is the load on the fiber \( i \), which depends on the external load \( F \) and on the state of the other fibers. The dynamics is now that each "spin" follows the local field (i.e., it breaks when \( F_i > x_i \)).
In the case of ELS fiber bundles, the load on each fiber is given by $F_i = 2F/(N = \sum_j s_j)$, while for LLS $F_i$ is a complicated function of the state of the neighboring fibers: if a fiber is close to a crack of length $k$ it carries a load of $F_i = F(1 + k/2)$. The crucial difference between these load sharing rules and ferromagnetic interactions is that for fibers the increase of the effective field due to other fibers breaking grows as the fracture process progresses, while in the RFM this quantity is always given by $2J$. For instance in the case of ELS, the breaking of a single fiber increases the effective field of the others by a quantity $\Delta F = F/(n(n-1))$ which increases as the fraction of intact fibers $n$ decreases. Similarly in the case of LLS, each time we break a fiber, we increase the effective field of the neighbor by a quantity $\Delta F = kF/2$ which becomes larger as more and more fibers break.

This difference has a profound effect on the phase diagram, because it implies that interactions always prevail over disorder. Thus, as the system size increases, the system will necessarily hit the low disorder first-order transition line.

We now turn our attention to more complicated lattice models, such as the random fuse model. The attempts made in the past to map fracture models into spin models are typically based on the lattice Green function formalism: removing a bond $i$ in the lattice implies that the load in the other bonds has to be changed by a quantity $G_{ij}$ (the Green function). Thus we can rewrite the effective field in terms of $G_{ij}$ and obtain a suitable spin model. For instance, in the random-fuse model $G_{ij}$ can be approximated as the dipolar kernel, but only in the limit where only a few bonds are broken. It is interesting to remark that in these conditions the Green function decays as $r^2$ in two dimensions, which is exactly the crossover point between local and global behavior in the RFM. This could imply that the spinodal point, inaccessible under LLS, could be reached in the RFM.

One should take these considerations with due care, since for a generic damage configuration, finding $G_{ij}$ amounts to solving the full problem (i.e., the Kirchhoff equations for the random fuse model) and is thus practically impossible. Thus we can rigorously map fracture models into spin models only in the dilute limit, which is not, however, the limit of interest in fracture. Extrapolating the results of spin models outside their regime of validity leads to inconsistent results, because the stress amplification at the crack tips is not properly captured by the dilute Green function. For instance, the authors of [78] treat in mean-field theory the spin model obtained from Green function formalism. Neglecting the load amplification in their analysis (i.e., the load transfer is chosen to be independent of the state), they obtain a spurious phase diagram similar to that obtained for the RFM. Similar problems affect the analysis presented in [77].
4.2 Percolation and fracture

4.2.1 Percolation scaling

Adding randomness to brittle lattice models of fracture can be done in many ways, the simplest being the random removal of a fraction of the elements. This process is best understood using the concept of percolation, a second-order phase transition that exhibits the full spectrum of usual characteristics, such as scaling and critical exponents as discussed in section §4.1.3 [78].

A percolation transition signifies that a structure becomes connected for certain values of the control parameter (i.e., the fraction of intact bonds). The usual example, demonstrated in figure 4.1, is that of bond percolation in a 2D square lattice, where a fraction $p$ of the bonds have been removed. The question is now whether the remaining ones form a connected (or spanning) cluster that extends through the lattice, connecting for instance the two horizontal edges. In this particular example, the spanning cluster is found, in the limit of large lattices, for $p < p_c = 1/2$.

As in other second-order phase transitions, the location of the critical point (i.e., the value of $p_c$) depends on the type of lattice, but the critical exponents are universal.

An important quantity for fracture in the percolation context is the order parameter, usually defined as the probability $\Pi$ that a bond belongs to the spanning cluster. Above the percolation
threshold $p_c$, the order parameter scales as

$$\Pi \sim (p - p_c)^\beta,$$  \hspace{1cm} (4.6)

with $\beta = 5/36$ in $d = 2$, while $\Pi = 0$ for $p < p_c$.

The essential part of describing percolation as a second-order phase transition is that there is a divergent correlation length $\xi$ scaling as

$$\xi \sim (p - p_c)^{-\nu},$$  \hspace{1cm} (4.7)

with $\nu = 4/3$ in $d = 2$. This correlation length can be measured analyzing the statistics of the clusters of connected bonds for $p < p_c$. The cluster radius distribution is a power-law up to a cutoff lengthscale $\xi$. For lengthscales larger than $\xi$ (for the system size $L \gg \xi$ or for distances between two points $r \gg \xi$) the system is out of the critical regime and is either connected or disconnected. For scales such that $r < \xi$, the physics of percolation is governed in general simply by the singly connected or red bonds. They are — in the bond percolation language — those that make the spanning cluster connected, so that removing any of them also destroys connectedness, or cuts the continuous, spanning path across the sample into two. The sensitivity to the lattice to small perturbations, like the removal of a red bond, is a signature of criticality.

Red bonds also play a crucial role on various important properties. For instance, if the bonds are taken to be fuses in a RFM, then red bonds will all carry the same current. Thus the conductivity (or equivalently the elastic stiffness in tensorial models) and strength are going to relate to the properties of such bonds. This allows to write down in the critical regions the scaling Ansatz for the conductivity

$$\Sigma \sim (p - p_c)^{\nu_c} f_{\text{perc}}(\xi/L^{1/\nu})$$  \hspace{1cm} (4.8)

such that for $L \gg \xi$ or $p$ large enough but still close to the critical regime, the scaling function $f$ becomes a constant.

### 4.3 Extreme statistics for independent cracks

In the presence of many non-interacting defects, it becomes apparent that the physics is related to extreme statistics (see figure 4.3). From the theory of fracture mechanics one usually collects the most important characteristics about the behaviour of the elastic fractures in a disordered medium, in the following three basic assumption [79].

1. **Non-interacting defects hypothesis**: defects are distributed throughout the medium in such a way that every defect is contained in a volume element, the strength of which it determines. We assume that no interaction between defects exists, so that the effect of the defect in each volume element can be analyzed, independent of that in any other element.
CHAPTER 4. STATISTICAL THEORY FOR FRACTURE MODELS

Figure 4.2: An example of a dilute defect population. These cracks each of some size \( l \) are assumed in the simplest case to have no influence on each other, and they have a linear size distribution \( P(l) \).

2. **Largest-crack hypothesis**: the strength of a volume element is related to the severity if the flaw it contains by Griffith’s theory of crack instability [1], so that there exists a critical defect size below which the system do not fails.

3. **Weakest-link hypothesis**: the strength of the whole material is uniquely defined by the strength of that volume element that contains the most severe defect.

The first two assumptions determine the distribution of local strength throughout the medium in terms of the distribution of flaws.

Starting from these basic assumption it is possible to derive a set of expressions for the distribution of the defect size and the distribution of the strength of the medium. Consider a sample of size \( L \). The “most severe defects”, that can carry the greater stress enhancement, can belong to one of two different family of defects that we define as type I and type II.

- **type I defect**: a single defect of length \( l \): the stress enhancement at the tips is \( \sim l^{1/2} \);
- **type II defect**: two neighbour defects of length \( l/2 \) each: the stress enhancement in the middle is \( \sim l \).

From this one can easily calculate that a pair of type II defects of length \( \sqrt{l}/2 \) conveys the same stress enhancement of a single defect of length \( l \). On the other hand, the appearance of type II defects is not guaranteed, depending on the topology of the system and the spatial distribution of the defects.

From Griffith theory, the failure stress should scale as

\[
\sigma_f \approx \frac{1}{l_{\text{max}}} \approx \frac{1}{\ln(L)}.
\]  (4.9)
given that, for large $L$, the size of the critical defect scale as

$$l_{\text{max}} \approx -\ln(L)/\ln(1-p). \quad (4.10)$$

Here the important exponent $\mu$ can lie in the range $[1, 2]$. If a pair of type II defects dominate the critical defect problem, then $\mu = 1$, while when isolated defects dominate the critical defect problem, then $\mu = 2$.

The critical stress is determined by the size $l$ of the largest defect in a sample. We define $C_L(l_{\text{max}})$ as the probability to find no defect larger than $l_{\text{max}}$ in the sample volume $L^d$. We can subdivide the volume into $(L/L_1)^d$ subvolumes of linear dimension $L_1$. If the characteristic size of the largest defect is much smaller than $L_1$, the subvolumes are independent and it follows that the probability to find no defects larger than $l_{\text{max}}$ on the whole sample is

$$[C_{L_1}(l_{\text{max}})]^{(L/L_1)^d} \approx C_L(l_{\text{max}}). \quad (4.11)$$

From the percolation theory \[78,80,81\] we known that the distribution for percolation cluster sizes scale, for large $l$, like

$$C_L(l) \approx 1 - cL^2 \exp(-kl). \quad (4.12)$$

Solving (4.11), we arrive \[17,38,82\] at the expression for the distribution of the largest cluster size

$$C_L(l_{\text{max}}) = \exp[-cL^2 \exp(-kl_{\text{max}})]. \quad (4.13)$$

This distribution is the Gumbel distribution and is appropriate as long as $p \ll p_c$. If otherwise the probability of finding a defect of size $l$ is an algebraically decaying function

$$C_L(l) \approx 1 - cL^d l^{-m}, \quad (4.14)$$

then the corresponding distribution for the largest size is

$$C_L(l_{\text{max}}) = \exp(-L^d l_{\text{max}}^m), \quad (4.15)$$

that is the well known Weibull distribution \[3,83\].

Using (4.13) we can obtain the distribution for the critical stress. By means of the dependence \(4.9\) between stress and largest size, the failure stress distribution function $F_L(\sigma)$ is

$$F_L(\sigma) = 1 - \exp\left[-cL^d \exp\left(-\frac{k}{\sigma}\right)\right]. \quad (4.16)$$

$F_L(\sigma)$ is the probability that a sample of size $L$ will fail if an external stress $\sigma$ is applied to the sample. The constant $k$ and $c$ vanishes at the rigidity percolation threshold.

The equivalent Weibull distribution for the critical stress is

$$F_L(\sigma) = 1 - \exp(-cL^2 \sigma^m), \quad (4.17)$$
where \( m = m'\mu \).

Another important quantity is the average failure stress. Duxbury and collaborators [47, 58, 82] derived an approximate expression by the assumption that this will be close to the median value \( \sigma_{1/2} \), obtaining for the Gumbel distribution

\[
\sigma_{1/2}^\mu = \frac{1}{A(p) + B(p)\ln(L)}, \tag{4.18a}
\]

\[
A(p) = \frac{\ln(c) - \ln[\ln(2)]]}{k}, \tag{4.18b}
\]

\[
B(p) = \frac{2}{k}, \tag{4.18c}
\]

and for the Weibull distribution

\[
\sigma_{1/2} \approx L^{-2m}. \tag{4.19}
\]

[3, 56, 84] [4, 57] [5, 57, 38, 82]
Disordered quasi-brittle material where a progressive accumulation and evolution of damage leads to failure has been object of intensive studying by the means of various discrete lattice models \([5, 59, 85]\); electrical fuse/breakdown models \([7, 53, 7, 84, 86, 87]\), central-force models \([2, 29, 38, 60, 67, 88, 89]\), bond-bending models \([50, 68]\), and beam-type models \([4, 59]\). In those models the disorder has been introduced in the elastic constants, threshold values or in the random dilution of the bonds. Exhaustive reviews can be found in \([7–9, 88]\) and references therein.

The essential features of discrete lattice models are disorder, elastic response characteristics, and a breaking rule for each of the bonds in the lattice. In this study, we consider the diluted disorder random fuse model. This is usually referred as a weakly disordered model, due to the fact that only a fraction of the system is involved, and can be obtained as a limiting model for the strong disordered random threshold fuse model, when the distribution function is symply the sum of two delta function, centered in 0 and 1. Consider a network of fuses, each having a resistance of \(1\Omega\) and a breaking threshold \(t_{ij} = 1\ A\ (1\ V)\). The characteristic of each fuse is linear until it reaches the breaking threshold, above which it becomes irreversibly an insulator. The underlying topology of the network is a lattice with cylindrical periodic condition, whose bonds are laid on the fuses. A fraction \(n_r\) of the fuses is then randomly removed by the system. The network is a conducting network as long as the fraction of bond removed is below the percolation threshold \(p_c\), since above that value it exists a connected path of fuses across the lattice. Now we apply two bus bar to the top and bottom side of the network, connected to an external potential difference \(V\). A current \(I\) will then flow through the network and, solving the linear set of Kirchhoff equations for the system, we are able to calculate the current \(i_{i,j}\) flowing in each fuses. If the current inside one of the fuses reaches the breaking threshold, the fuse will burn. If, for a given applied potential, the network is still conducting (i.e. if there exists a percolating cluster of intact fuses) we increase the potential until a new fuse fails. The evolution of the system will then be a succession of avalanches, a set of fuses that automatically burn in sequence.
when the voltage is raised, until one last avalanches disconnects the network. All the steps are executed adiabatically, assuming that the reallocation of the current inside the system is much faster than the timescale of the voltage increase.

The simulation software follows almost exactly the steps given in the description of the model. We initially create a network of fuses and then we remove a fraction of them. The system of linear equation obtained considering the potential is then solved in matricial form. The resulting stiffness matrices are sparse matrices, allowing us to perform a Cholesky decomposition on them. Once solved, we are able to search for the fuse that will break. To speed up the computational tasks, we maintain fixed the external potential difference to $V$ and search for the fuse with the highest current (since the resistance is $1 \Omega$). As the system is linear we are able to simply calculate back the actual value of $V$ and $C$ needed to really break the fuse. When the solution shows that no more current is flowing through the busbars, we known that a spanning cluster of insulators has disconnected the network.

5.2 Simulation Details

Fracture simulations are performed mainly on diamond-like lattice, with disorder $n_r = 0.10$ and $n_r = 0.30$. We simulated samples sizing $2^l$, where $l$ ranges from 3 to 10. For every set of the parameters we simulated $10^5$ different realizations of the disorder, except for $L = 2^{10}$ where, due to memory limitation, we performed only 20,000 realizations.

In addition we also realize a small set ($10^4$ realization for each size) of simulations with a square topology, limited to only $n_r = 0.10$.

5.3 Observables of interest

The most important quantities in the study of the fracture mechanics are the stress and the strain, corresponding in our model to the linear applied current and voltage. During the simulations we collect them at the first bond breaking, at the point of maximum voltage and at the point of maximum current. Another important quantity is the number of fuses burned in order to reach the various steps, corresponding to the amount of damage accumulated by the system. In addition we also record the ordered list of the fuses burned, allowing us to reconstruct the cluster evolution and analyse the size of the defect clusters. Hereinafter we will refer to the point of maximum current as the peak load and we will use the notation $C_{\text{max}}$ (e.g. for the subscripts); similarly for variables relative respectively to the point of maximum voltage and the first bond breaking we will use the notation $V_{\text{max}}$ and $1\text{bb}$.

There exists a formal equivalence between the random fuse model current-driven or voltage-driven, involving the computation of the lattice conductivity: the current flowing through the
system can be written as
\[ I = \Sigma V, \tag{5.1} \]
where \( \Sigma \) is the global conductivity
\[ \Sigma = \frac{\sum_{ij} \sigma_{ij} V_i^2}{V^2}. \tag{5.2} \]

In the rest of the work we will ever refer to the model as being *current-driven*.

To highlight the connection with the elasticity (as exposed in section §5.1.4), we recall that to apply a given linear density of voltage amounts to impose a displacement to an elastic medium, while maintaining a fixed current through the system is equivalent to impose a fixed load. On the basis of this analogy, we proceed to define the stress as the linear density of current flowing over the bars \( \sigma \equiv I/\mathcal{L} \) and the strain as the corresponding linear density of voltage difference \( \epsilon \equiv V/\mathcal{L} \), where \( \mathcal{L} \) is the linear size of the top and bottom of the system (\( \mathcal{L} = L \) for square systems and \( \mathcal{L} = 2L \) for diamondlike systems, see figure 5.3).

### 5.4 Relations between stresses and strains at various timesteps

Early in the literature it was common usage \[17,57,84\], based on the weakest-link hypothesis, to relate the definition of the fracture strength, to the stress or the strain required to break the very first bond. This can prove to be inadequate. It is known for instance that in the random threshold fuse model (see e.g. \[53\]) the weakest-link hypothesis doesn’t hold not even at the peak load: even in the postfailure regime the accumulation of the damage remains diffusive and damage localization happens (by coalescence of microcrack) only in the final stages. In the diluted disorder model that we are dealing with, the weakest disorder doesn’t allow damage diffusion, — a spanning crack actually appears — but one still has to require the validity of the

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*Figure 5.1: Square (left) and diamond (right) topology. Dashed bonds lie on the periodic boundary. Note that the two lattices, both composed by 8 × 8 sites, have different bond size of the bases: \( \mathcal{L} \) is equal to \( L \) for the squares and to \( 2L \) for the diamonds.*
weakest link hypothesis. The fact that the first fuse that break does not cause the complete failure can have strong consequences on what happens at the peak load. From the comparison between the main quantity at the first bond breaking and at the peak load, showed in figure 5.2, the damage accumulated at the peak load (the number of broken bonds) scales as a power law. The evolution of the initial distribution of defect clusters is driven not only by the weakest cracks in the system, but also by the enhancement and shielding of the stress around the defects. Similarly the cluster distribution at the peak load will be different from the initial distribution and then a similar behaviour may be expected for the critical stress.

![Figure 5.2: Comparison of the current and the density of broken bonds at the first bond breaking and at the peak load.](image)

### 5.4.1 Coalescence between the point of maximum current and voltage

As one can see in figure 5.3, the difference between the value of the current, and hence the stress, at $C_{\text{max}}$ and at $V_{\text{max}}$ is a fast decreasing function. For low-disordered systems, one can say that yet at $L = 64$ the two values are indistinguishable. Increasing the disorder however results in a big shift of this behaviour and, if $n_r = 0.30$, only for the largest systems $L = 1024$ one can safely switch the two variables. In this case is noteworthy that the difference $(I_{V_{\text{max}}} - C_{\text{max}})$ increases when $L$ goes from 8 to 32; we will see in many instances that the sizes ranging up to 64 are prone to finite-size deviations (see chapter Chapter 8.2). The distance in the number of broken bonds (on the righth side in figure 5.3) is also decreasing, with a overextensive scaling ($\sim L^{-4}$). We want here give a first note about the relations among the systems possessing different topologies or disorder. If $n_r = 0.10$, the square and the diamond samples behave almost equally, while systems with more disorder differs only for a scale factor.
5.4. RELATIONS BETWEEN STRESSES AND STRAINS AT VARIOUS TIMESTEPS

5.4.1 finite size effects. Unfortunately, there exists serious finite size effects that affect “small” system sizes at the maximum of the voltage. This effects are due to the winding of the (future spanning) crack around the periodic direction of the sample. Due to this behaviour, the data presents a very large amount of finite-size corrections and they are not easily understandable. An exhaustive study about this problem will be given in the chapter 8.2.

5.4.2 stress and strain

The random fuse model is actually a semi-brittle model, as one can guess since the peak load never appears at the first bond breaking. This is indeed an indicator of damage accumulation prior to the failure. This fact becomes more clear analysing the time behaviour of the stress and the strain, as showed in figure 5.3. The lifespan of both can be divided in three different zones: a first and a last zone when they are quickly decreasing and possess a similar behaviour, and a central zone where the slope of the strain is less pronounced with respect to the stress. A possible explanation can be given in relation to the fracture growth: when the crack is small or when its two extremities are next to close even a small change in the length induces a large difference in the distribution of the strain. On the other hand, in the meantime the crack is enough large that the widening of a single bonds induces only a small relative increase in the size and the two extremities are too far to interact.

Figure 5.3: Comparison between the points of maximum current $C_{\text{max}}$ and of maximum voltage $V_{\text{max}}$. In the thermodynamic limit, the two point occurs at the same time and the observables take the same values. The diamondlike systems and the square system, being the disorder equal, behave almost identical. Increasing the disorder induces an increase in the differences, but the slope of the convergence (5.3b) remains the same.
Figure 5.4: Current-voltage diagram for a single sample $L = 256$. In the inset — showing the earlier times —, the current peak and the voltage peak are clearly visible. The following points belong to the post-failure regime. Paths going right-to-left correspond to voltage-driven avalanches, paths top-to-down correspond to current-driven avalanches.

Figure 5.5: Stress and strain timelines. Strains are represented by darker points and stresses by lighter ones lines. Here, for every $L$, $\sigma^*$ and $\epsilon^*$ are the reduced stress and strain ($\sigma^* \equiv \sigma/\sigma_{\text{max},L}$ and $\epsilon^* \equiv \epsilon/\epsilon_{\text{max},L}$) and the time is the reduced time ($t^* \equiv N/N_{b,L}$). Stress and strain at a given time $t$ are average over all samples. Inset: an enlargement of the upper left angle (earlier times) showing the peak of the applied stress in a $L = 512$ system; axis as in the main figure.
5.5  The process of the fracture

In the next chapters, we analyse in detail the processes that drive the failure of the random fuse model. This process can be divided in four timesteps:

- the first bond breaking at the beginning;
- a damage accumulation transient period;
- the peak load;
- the postfailure regime.

While in the two transient periods — the damage accumulation prior to the peak and the subsequent postfailure avalanche — we are mainly interested in the dynamical aspects, such as the behaviour of the expanding fracture, at the beginning and at the peak load we are interested mainly in the "frozen" properties of the system, such as the actual distribution of the clusters width or the strength.

5.5.1  statistical distributions for disordered systems

Let us recall the main theoretical points about the statistics of disordered systems (for a more exhaustive exposition, see section §7.6). We assume that in an elastic disordered system the following conditions hold:

- the defects are independent (non-interacting defects hypothesis);
- the system failure is ruled by the strength of the "most critical" defect (Weakest-link hypothesis);
- there exist a critical defect size (Largest-crack hypothesis).

Under this assumptions, if the cluster distribution decays exponentially or algebraically (as expected if we are far or near the percolation threshold), the distribution of the largest cluster is, respectively,

\[ C_L(l_{\text{max}}) = \exp[-\rho L^d \exp(-kl_{\text{max}})], \quad \text{(5.3a)} \]
\[ C_L(l_{\text{max}}) = \exp[-\rho L^d l_{\text{max}}^{\mu}], \quad \text{(5.3b)} \]

Since the relation between stress and largest size reads \( \sigma \sim l_{\text{max}}^\mu \), the failure strength \( S \) of a material is described by

\[ S_L(\sigma) = \exp[-\rho L^d \exp\left(-\left(\frac{\sigma_0}{\sigma}\right)^\mu\right)], \quad \text{(5.4a)} \]
\[ S_L(\sigma) = \exp[-cL^2 \sigma^m]. \quad \text{(5.4b)} \]
where \( d \) is the dimensionality of the system, \( m, \rho \) and \( \sigma_0 = k^{1/\mu} \) parameters. The parameter \( \mu \) is bounded in \([1, 2(d-1)]\) and depends on the actual topology and the distribution of the defects inside the lattice. If the stress enhancement is completely determined by single long defects (type I defects), \( \mu = 2 \); if it is determined by couples of near-touching defects (type II defects) \( \mu = 1 \). Using energetic arguments, one obtain the inequality \([1, 17, 60]\)

\[
1 \leq \mu_1 \leq \mu_b \leq 2(d-1)
\]  

(5.5)

where \( \mu_1 \) is related to the defect that cause the most current in a bond and \( \mu_b \) to the defect that breaks the system. This distributions are not supposed to hold only at the peak load but at different times the actual distributions are expected to be different. For instance, in the case of Gumbel distribution, one expect \([17]\) that \( \mu \) may have different values at the first failure and at the peak load.

In our analysis we mainly use of the Gumbel equation \([5.44]\), therefore we want to state clearly the role played by the two exponents \( d \) and \( \mu \): the former is derived from the hypothesis that the material can be divided in non-interacting subvolumes. If \( d \neq 2 \), our case, it means that this hypothesis is not entirely satisfied, and the interactions between the cracks are not completely negligible. The latter depends on the geometrical aspects of the stress enhancement, so obviously it is affected too by the independance of the cracks.

The discriminator between the two strength distribution is the distribution of the defect widths inside the material: if it is exponential, one obtains the modified Gumbel distribution for some values of the parameter, if the clusters are distributed according to a power-law, one expect to retrieve the Weibull distribution.

The mean failure strength: There is not in the literature an established formula for the mean failure stress, although there exists some agreement (in both theory and experiment) that its scaling behaviour depends on the logarithm of the size as a power law \( \sigma^{\mu} \sim 1/\ln(L) \). A first explicit expression is due to Duxbury and collaborators \([17, 58, 62]\) that introduced the median \( \sigma_{1/2}^{\mu} \) of the (modified) Gumbel distribution \([5.44]\) as the an approximate expression assuming that this will be close to the mean value.

\[
\sigma_{1/2}^{\mu} = \frac{1}{A + B \ln(L)}.
\]  

(5.6)

where

\[
A = \frac{\ln(c) - \ln[\ln(2)]}{k}, \quad \text{and} \quad B = \frac{d}{k}.
\]  

(5.7)

Usually a slightly different expression is commonly used substituting \( \ln[\ln(2)] \) in \((5.7)\) with a generic constant \( a \).

Unfortunately, it is not possible to exactly know how much “close” one this expression is to the “real” mean. The strictest bond between mean and median one can state is a consequence of
5.5. THE PROCESS OF THE FRACTURE

the Chebishev inequality \[ |\mu - m| \leq \sigma. \] (5.8)

We want here to demonstrate that the formula (5.6), that we rewrite as

\[ \bar{\sigma} = \frac{\frac{1}{\sigma^\mu}}{\left[d \ln(L) + \ln(\rho) - \ln(-\ln(\chi))\right]^{1/\mu}} \] (5.9)

represents the exact expression for the mean fracture strength at least in the large \( L \) limit, when the Gumbel distribution (5.44) holds. Consider the cumulative distribution function \( S_L(\sigma) \) for the variable \( \sigma \),

\[ S_L(\sigma) = \exp\left[-\rho L^d \exp\left(-\frac{k}{\sigma^\mu}\right)\right], \] (5.10)

and derive a general formula to evaluate the stress \( \sigma_X \) such that \( S_L(\sigma_X) = \chi \) — this is the quantile function —:

\[ S_L(\sigma_X) = \chi \quad \Rightarrow \quad \exp\left[-\rho L^d \exp\left(-\frac{k}{\sigma^\mu}\right)\right] = A \quad \Rightarrow \]

\[ \Rightarrow \quad \frac{k}{\sigma^\mu} = d \ln(L) + \ln(\rho) - \ln(-\ln(\chi)) \quad \Rightarrow \]

\[ \Rightarrow \quad \sigma_X = k^{1/\mu} \left[d \ln(L) + \ln(\rho) - \ln(-\ln(\chi))\right]^{1/\mu} \]

If there exists \( X \) such that \( S_L(\sigma_X) = \chi, \forall L \), then \( \sigma_X = \chi \) and (5.9) is true.

This is exactly what happens for the (not-modified) Gumbel distribution, where

\[ F(x) = A \quad \Rightarrow \quad x_A = \mu - \beta \ln(-\ln(A)), \] (5.11)

if \( F(x) \) is the cumulative Gumbel distribution in the most usual formulation with location and scale parameters \( \mu \) and \( \beta \). Here, for every values of the parameters, the formula (5.11) provides the median when \( A = 1/2 \) and the mean when \( A = \exp(-\exp(-\gamma)) \), where \( \gamma \) is the Euler-Mascheroni constant \( \gamma \sim 0.577 \).

As recently demonstrated by Shekhawat [92], \( S_L(\sigma) \) converge asymptotically to the Gumbel distribution in the large \( L \) limit. Then, at least in this limit, equation (5.9) provides the expression for mean strength.

When working with raw data, one deals with the sum of the parameters \( \chi \) and \( \rho \), that are both unknown. Then we usually rewrite eq. (5.9) as

\[ \sigma_0 = \frac{k^{1/\mu}}{\left[d \ln(L) + \ln(\rho_0)\right]^{1/\mu}}, \] (5.12)

where \( \rho_0 = -\rho/\ln(\chi) \) and, for internal notational consistency, \( k^{1/\mu} = \sigma_0 \).
We believe anyway that the formula \(5.3\) is valid for every value of \(L\), i.e. we believe that the requirement we showed valid for the Gumbel distribution may be valid for every distribution, under some regularity requirements. We have now introduced a new formula for the mean stress but, knowing that this old formula represents the exact expression makes available an alternative way to evaluate the distribution parameters and the actual scaling of the data. We note that, when used in its original formulation, the median \(\sigma_{1/2}\) induce an error due to actual difference between 1/2 and \(\xi\), for instance for the data we collect at the peak load, while the median \(\sigma_{1/2}\) by definition correspond to \(S_L(\sigma_{1/2}) = 0.5\), the mean \(\bar{\sigma}\) is such that \(S_L(\bar{\sigma}) \sim 0.62\). This fact can obviously introduce deviations in the analysis.
Six

The Quasi-Brittle Phase

Just before the first bond breaking, when the current is off, the state of the system uniquely depends, by construction, on the way we prepared it. This is the main difference between this state and the peak load, and also between this model and a real world material. In all other cases, some external force has already acted on the medium, and the internal microcracks have reacted to this force, changing their distribution.

6.1 Largest crack and weakest link hypotheses

![Graph](image)

Figure 6.1: Fraction of the broken clusters that are the largest ones before the breaking ($P_I$), that becomes the largest ones after the breaking ($P_{II}$) and that will belongs to the failure percolating cluster ($P_{perc}$). [big-max_1bb]

The largest crack hypothesis assumes that the most critical defect is the largest cluster (defect of type I, see section §6.3). Actually it is possible that the most stressed bond is a defect of type
II (two clusters separated by a single bond). To have a clue, we study fraction $P_I$ of the clusters that were the largest ones before the fuse-break. Another interesting quantity is the fraction $P_{II}$ of clusters that were not the largest ones but becomes so after the fuse-break: this is a good indicator of the appearance of a type II defects, but represents only a sufficient condition: the fuse that fails can belong to a type II defects whose final size is still not the largest. The last quantity we study here is the probability $P_{perc}$ that the fuse broken will belongs to the final fracture, i.e. the probability that the crack started here propagates until percolation. The shape of cracks is highly linear, corresponding to a single-bond-width line with only rare bifurcations. Then, in order to check the largest crack hypothesis and to ensure comparability with other works, we define the width of a crack as the end-to-end horizontal length.

The probability that the largest crack hypothesis is verified is the probability that the broken bond is the largest and belongs to the final spanning cluster, i.e. the product of the probability to be the largest cluster and to percolate:

$$P_{lch} = P_I \sim L^{-0.75} P_{perc} \quad (6.1)$$

What emerges is that both $P_I$ and $P_{perc}$ are decreasing with the size, with a similar behaviour. At the opposite, $P_{II}$ is increasing with $L$, in a way such that $P_I + P_{II}$ is almost constant and very close to unity. Therefore almost always the breaking cluster will be, after the bond breaking, the largest one; this is due to the presence of type II defects. The coupled scaling of $P_I$ and $P_{perc}$ signals that, when it happens that the largest crack starts to break, it will propagate until failure, proving the assumption of the initial hypothesis. On the other hand, the behaviour of $P_I$ is somewhat predictable given that the average number of steps needed to reach the peak load grows as a power-law: we known now that this is not due to the arrest of the expanding largest crack. The decrease of $P_I$ depends also on the fraction of bonds that sizes the largest with respect to the whole bond population. As we will show in the following section §6.3, this fraction is exponentially decreasing.

### 6.2 Crack cluster width distribution

In the following, the crack-cluster distribution and the largest crack distribution are examined at the peak load. Characterization of these distributions is important since these distributions determine the type of distribution followed by the fracture strength [4, 8, 13].

The distribution of the sizes of the crack clusters at the beginning is uniquely determined by the way we put the disorder in the model. On a general basis it is possible to compute it analytically, but this is not always a feasible task. More often the preferred way is to identify it by means of an “a posteriori” data analysis. In our case from the percolation theory [83, 85] we know that, far away from the percolation threshold, the cluster distribution is exponential. The fraction $n_r$ of bond we removed should put us enough “far” from the threshold.
6.3. **First Failure Stress Distribution**

6.3.1 **Gumbel and Weibull tests**

From the extremal theory, the exponential scaling \((6.2)\) of the clusters implies that the strength and the largest size distribution are described, respectively, by the Gumbel-like forms \((5.4a)\) and \((5.3a)\). The usual way to check if some data follow a given law, is to plot the data in such a way that they should form a straight line. In figure 6.3, we plot the data both as \(1/\sigma^2\) versus \(\ln[-L^{-2}\ln(S_L(\sigma))]\) and as a double log plot of \(\sigma\) versus \(L^{-2}\ln(S_L(\sigma))\), the former resulting in a Gumbel test plot and the latter in a Weibull test plot. This type of test is working well only in the case when one knows in advance the exact value of the constant \(\mu\). For the test we adopted the value \(\mu = 2\), the one resulting if the stress enhancement is entirely dominated by type I defects. What in fact results from the visual inspection of figure 6.3 is that the data surely don't obey to
the Weibull law, while the Gumbel test, even though well behaving, presents a tiny curvature. There is indeed the need to search for another way to obtain a more correct value of $\mu$.

We can find the correct value of $\mu$ via the scaling of the mean stress $\sigma_{1bb}$. If the underlying distribution is the Gumbel-like form of eq. (5.4a), namely

$$S_L(\sigma) = \exp\left(-\rho L^2 \exp\left(-\left(\frac{\sigma_0}{\sigma}\right)^\mu\right)\right),$$

then the mean stress has the form:

$$\sigma_L = \frac{\sigma_0}{\sqrt{2 \ln(L) - \ln(\rho_0)}}^{1/\mu},$$

(6.3)

where $\rho_0 = \frac{\rho}{\ln(\chi)}$ and $\chi$ is a constant (see app. §5.5.1).

The resulting fit — showed in figure 6.4 — provides a value smaller than expected: $\mu = 1.484$. This is another confirmation of the lack of validity of the largest-crack assumption. Another information that can be extracted from the mean is the dimensional exponent $d$. The resulting $d = 2$ supports the hypothesis that the configuration of the clusters is enough sparse to make them non-interacting. Anyway, from the expression of the mean (the other parameters are $\sigma_0 = 1.452$ and $\ln(\rho_0) = 0.958$) do not allows us to reconstruct the complete form of the distribution. This is due to the fact that the $\rho_0$ is the sum of other two parameter $\rho$ and $\chi$ of which only the first enter in the Gumbel expression.
6.3. \textit{FIRST FAILURE STRESS DISTRIBUTION}

![Figure 6.4: Mean stress $\sigma_L$ at the first bond breaking. Its scaling confirm the scaling-effects as indipendents subvolume ($d = 2$) but not that of the largest-crack ($\mu = 1.484$). [stress_1bb_dia10]](image)

### 6.3.3 Size-effects

The scaling of the mean stress support the presence of size-effects depending on $L^2$, then in the terms of indipendents subvolumes, but this is actually effective up to small corrections. In figure 6.5a we plot, for every size $L$, the strength $S_L(\sigma)$ and $[S_L/2(\sigma)]^4$. They are expected to collapse in pairs. The small deviation present mostly in the smaller sizes is in the form of a shift correction to $\sigma$ that we was able to identify but that still lacks a full theoretical basis. Calculating the amount of shift $\sigma'$ needed to make the distribution $S(\sigma - \sigma')$ collapse in figure 6.5a, the corrections results (see fig. 6.6) a subextensive power-law shift $\sigma \rightarrow \sigma - \sigma'$, where

\[ \sigma' = 0.915L^{-1.419} \tag{6.4} \]

Using the shift-corrected distribution we are now able to make all lines collapse (figure 6.5b).
CHAPTER 6. THE QUASI-BRITTLE PHASE

Figure 6.5: Survival probability $S_L(\sigma)$ at the first bond breaking. Dashed lines are $S_{L/2}^4(\sigma)$. A shift correction (see eq. (6.4)) is needed to obtain full coalescence. [size_1bb_010]

Figure 6.6: Shift correction to the size-effects scaling behaviour of $S_L(\sigma)$. [shift_pars_1bb]
6.3.4 renormalization

Knowing now the parameters \(d, \mu, \rho_0\) and \(\sigma_0\) of the Gumbel distribution (5.43), we can renormalize the data as in figure 6.7 and separate \(\rho_0 = -\ln(\rho / \ln(\xi))\):

\[
\ln\left(-\frac{1}{L}\right) = \ln(\xi) + \frac{1}{\rho_0} \ln\left(\frac{\sigma}{\sigma^*}ight)
\]

We obtain

\[
\begin{align*}
\rho & = 2.342, \\
\sigma_0 & = 1.475.
\end{align*}
\]

From \(\rho\) we can calculate \(\chi = 0.407\).

6.4 Widest crack distribution

The cumulative distribution \(W_L^*(w)\) of the size of the largest clusters is expected to satisfy the Gumbel relations (5.39). As shown in fig. 6.8, a combined size-effects and Gumbel test makes all \(W_L^*\) collapse in a straight line. Analysing the data, one obtains that

\[
W_L^*(w) = \exp\left[-\rho L^2 \exp(-w_0 w)\right],
\]

where

\[
\begin{align*}
\rho & = 0.1317, \\
w_0 & = 0.628.
\end{align*}
\]

The scaling of the mean size of the largest cluster (figure 6.9) further confirms the values of the parameters and give

\[
\begin{align*}
w_0 & = 0.626, \\
\rho & = 1.132.
\end{align*}
\]
CHAPTER 6. THE QUASI-BRITTLE PHASE

Figure 6.8: Cumulative distributions $W_L^*(w)$ at 1bb: size-effects and Gumbel tests. [widest-cumul_1bb-fit3]

Figure 6.9: Mean size of the widest cluster $W(L)$. [medie_WL_1bb]
6.5 The Damage Evolution

Here we analyse the behaviour of the accumulation of damage at peak load \( C_{\text{max}} \).

The number of broken bonds at \( C_{\text{max}} \) is the accumulated damage. This is why we call "quasi-brittle" this model.

A quantity of practical interest is the number of broken bonds needed to reach the peak load. It not only represents the amount of damage that a material can sustain before breaking apart, but analysing its scaling, one can extract information about "how" this damage is localized. From the behaviour of related models, mainly the random threshold fuse model, we expect a subextensive power-law scaling that reflects a size-effect that decreases with the volume, \( N_{\text{tot}} \):

\[
N_c \sim N_{\text{tot}}^b,
\]

where the commonly found values are in the range \( b = 0.91 - 0.93 \) \([4, 69, 88, 83]\). The data for our dilute disorder model (displayed in figure 6.10) confirm the behaviour, with the difference that here the damage accumulation is much less, yielding an exponent \( b = 0.204 \). This difference can be explained by the amount and the different kind of disorder in the two models.

The mean number of cluster per sample \( M_{1bb} \) (fig. 6.11) due to the preparation is an extensive variable that reflects that way the disorder has been allocated. A most interesting quantity is \( M' = M_{1bb} - M_{C_{\text{max}}} \), the difference between the number of cluster present at the beginning and the number of cluster that one can count at the peak load. This is exactly the number of cluster disappeared in this period of time or, in yet other words, the number of times that the coalescence
of two crack has happened. According to the scaling of $M'$, we can see how the damage is stored. From the data (displayed in figure 6.12) we can't obtain a global scaling function but, at least for large $L$ ($L \geq 64$), they seem to scale as power-law $M' \sim L^b$ characterized an exponent $b = 0.635$ large, if compared to the number of broken bonds. Then the fraction of damage stored in type II defects is an increasing function.
The peak load is characterized by the minimum current needed to break completely the system, that is the maximum reached in the simulation. From this point onwards all subsequent broken bonds belong to the last avalanche. Contrary to the first bond breaking, when the internal structure was externally built, here it has been shaped by the damage accumulated in to the dynamical evolution of the model. If at the first bond breaking the distributions of the clusters were in theory analytically computable, here the accumulation of the damage is responsible for a fundamental complication in the evaluation of scaling and size-effects. There is even the possibility that, even in the diluted limit, the damage changes the asymptotic distribution \([16, 94]\).

**7.1 Largest crack and weakest link hypotheses**

In section §6.1 we showed that in the thermodinamic limit the bond that breaks don’t belong to the percolating cluster, even if in almost all cases it is the most critical one. On the exact opposite, the crack that breaks at the peak-load will belong almost surely to the future percolating cluster. This means that from here we have no phenomena of fracture arrest: unlike the random threshold model, there is no sparse damage diffusion, but just after the peak the damage localizes. We recover then the validity of the weakest-link hypothesis. The crack cluster that breaks, on the other hand, most likely is not the largest one. While the probability to percolate is next to the unity, the fraction of broken clusters at \(C_{\text{max}}\) that are the largest decreases with \(L\). It is noteworthy to analyse also the fraction of clusters that were not the largest ones before the peak, but after it. Previously we connected them with the appearance of type II defects; now at the peak load the coalescence of large defects is a not-so-common event.

**7.2 Crack clusters width distribution**

We discover now the effects of the damage accumulation: the cluster width distribution has become bimodal (see figure 7.2). It still preserve an exponentially decaying form, but now is
composed by two parts:

\[ \mathcal{W}_{\text{max}}(w) = \mathcal{W}'(w) + \mathcal{W}''(w), \]  

where both distribution are in the form \( \mathcal{W}(w) = \alpha e^{\beta w} \). Noteworthy, the two modes are well separated such that \( \mathcal{W}'(w) \) describes the distribution of the smaller cluster, while \( \mathcal{W}''(w) \) describes the distribution of larger cluster. From the analysis of the data the distribution of smaller sizes results almost identical to the distribution of the sizes at the first bond breaking, \( \mathcal{W}'(w) \approx \mathcal{W}_{\text{1bb}}(w) \), while the other one reveals a nontrivial size-dependence. We derived the distribution parameters \( \alpha \) and \( \beta \) individually for each one and we found that they satisfy the following relations:

\[ \begin{cases} 
\alpha = 11.72 L^{-1.43} \\
\beta = -(0.500 + L^{-7.14}). 
\end{cases} \]

Then, in the large \( L \) limit, the larger sizes scale as \( \mathcal{W}''(w) \sim e^{-0.5w/L^{1.43}} \). Why the damage seems to selectively affect only the distribution of the largest defects? This happens because, as dictated by energetic arguments, the wider are the cracks, the bigger is the probability they have to be further enlarged. If the appearance of a bimodal distribution is easy to understand, we lack of a full understanding of the scaling of \( \mathcal{W}''(w) \). Qualitatively may be the distribution of the screening in the internal structure that make the difference. Loosely speaking, in larger systems it is more difficult to find wide (respect to the whole volume) patchy zones.

Bimodal distributions of defects has yet observed, both numerically and experimentally \[ 53, 55-57 \] in fracture process. Anyhow the multimodal nature was there an intrinsic property of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.1.png}
\caption{Fraction of the broken clusters that are the largest ones before the breaking (\( P_1 \)), that becomes the largest ones after the breaking (\( P_{\text{H}} \)) and that will belongs to the failure percolating cluster (\( P_{\text{perc}} \)).}
\end{figure}
the materials, as in the case of pores and dislocations as defects. In these cases the strength distribution were also only representable with multimodal distributions.

![Graph showing distribution of cluster widths](image)

Figure 7.2: Distribution \( W_{\text{Cmax}}(w) \) of the widths \( w \) of the clusters at \( C_{\text{max}} \). The black line is the initial distribution \( W_{\text{1st}}(w) = 4.097e^{-1.6292} \).

We fit each single distributions \( W''_{\text{Cmax}}(w; L) \) to find out if there is a scaling law that rules the parameters (see figure 7.2). We found that, given the form \( W''_{\text{Cmax}}(w; L) = \alpha e^{\beta w} \), the parameters \( \alpha \) and \( \beta \) satisfy the following relations (see figures 7.3):

![Graphs showing scaling laws](image)

Figure 7.3: Size scaling for the parameter of the new cluster distribution.
CHAPTER 7. THE BREAKDOWN

7.3 Largest crack distribution

We then analyse the distribution of the size of the largest clusters, to obtain informations on how the branching in the cluster size scaling law has affected their distribution. Now the scaling (showed in figure 7.4) is not in agreement with a Gumbel distribution (6.5) in the same form than the initial distribution. The deviation is however not very large — there survives a resemblance with a straight line and also a sort of parallelism — and there is no clear sign of a multimodal distribution. This led us to propose a new form of the Gumbel distribution, along the lines of that for the strength, relaxing the requirement that the size of cluster must appears at the first power.

\[ W_L = \exp \left[ -\rho L^d \exp((-w_0 w)^\mu) \right]. \] (7.2)

Before confronting the data with this new form, we need to analyse the scaling of the mean size in order to know the correct value of \( \mu \) to use. The data agree well with the expression of the mean size

\[ w = w_0 [d \ln(L) + \rho_0]^{1/\mu}, \] (7.3)

as showed in figure 7.5, yielding the new value for the dimensional parameter \( d \) and scalin parameter \( \mu \)

\[ \begin{aligned} d &= 1.4877, \\
\mu &= 1/1.2971 = 0.771. \end{aligned} \]

Using these values to scale again the data, the new form \( 7.2 \) appear to be able to better fit the data than the older one. However, in figure 7.6, the collapse among the data is not perfect, but it
becomes more fitting, taking in consideration only the larger sizes. This sound reasonable, given the size-dependent scaling of $W''(w)$. We believe that, up to some correction decaying in $L$, the distribution of the largest cluster is represented by the form [7.2].

Figure 7.5: Mean size of the widest cluster $W(L)$.

Figure 7.6: Cumulative distributions $W'_L(w)$ at $C_{\text{max}}$: size-effects and Gumbel tests with the new form.
7.4 Strength distribution

7.4.1 Gumbel and Weibull tests

We have seen that the cluster distribution is bimodal, though the separation of the mode is not sufficient to induce a separation in the distribution of the maxima, that largely belong to only one mode. The failure strength distribution appear to be even less sensible to all of this: the scaling of the data is in very good agreement with the proposed distribution \( (\text{5.44}) \), where the values of the parameters are the “classical” ones \( d = 2 \) and \( \mu = 2 \).

We make a tests for the Weibull distribution too. In this latter case, as we see in figure 7.7, the matching is very poor — the lines are noticeably curved — while the line in the Gumbel plot are very straight. Only the tails present an irregular behaviour; we have however to say that we don’t known if they doesn’t fit well or if this is due to a lack of statistics.

![Modified Gumbel and Weibull test plots. If the data would obey a distribution, they should collapse into a straight line.](image)

7.4.2 Mean failure stress

The correspondence between the theoretical expectations and the model can be controlled, studying the scaling of the mean failure stress \( \sigma_L \). If the underlying distribution is the modified Gumbel one (eq. \( \text{(5.44)} \)), then the mean stress has the form:

\[
\bar{\sigma}_L = \frac{\sigma_0}{\left[ 2 \ln(L) + \ln(\rho_0) \right]^{1/2}},
\]

where \( \rho_0 = -\rho / \ln \chi \). The scaling (figure 7.8), is in very good agreement with our theoretical prediction. We want to point out that, before to derive the expression for the mean stress, we
compare these data with many different functions of power-laws and logarithm; none of them (usually depending on three parameters) has ever given a similar good agreement.

The parameters of the mean stress confirm the presence of size-effects $L = 2$ and the weakest-link arguments $\mu = 2$, as expected. We can also obtain the value of $\sigma_0 = 0.934$

![Figure 7.8: Mean stress $\sigma_{L}$ at the peak load. Its scaling confirm the scaling-effects as indipendents subvolume ($d = 2$) and the weakest-link hypothesis ($\mu = 2$).](image)

### 7.4.3 Size-effects

The results of the size-effects test, displayed in figure [7.9a], are in agreement with the hypothesis that $d = 2$, but also here, as yet at the first bond breaking, the curve collapse is not perfect due to the existence of a shift correction for the stress. We calculate the amount of shift minimizing the distance between the lines (figure [7.9b]) and we obtain a power-law corrective form

$$\sigma' \to \sigma - \sigma' \quad \text{where} \quad \sigma' = a + b/L.$$  \hspace{1cm} (7.5)

This corrections are decreasing as the inverse of the size while the tiny value ($0.00036$) of the constant $a$ is supposed to not represent a finite scaling limit, but rather an artifact introduced by the fact that our collapse brings all lines over the one relative to $L = 1024$, that is a finite size.

### 7.4.4 renormalization

In order to obtain the last parameter of the Gumbel distribution, we renormalize the data and fit them against a straight line (fig. [7.11]). They are in very good agreement with the scaling function $a - b/\sigma^2$. 

Figure 7.9: Survival probability $S_L(\sigma)$ at the peak load. Dashed lines are $S_{L/2}^L(\sigma)$. A shift correction (see eq. (7.3)) is needed to obtain full coalescence.

Figure 7.10: Scaling of the shift correction for size effects.
7.4. STRENGTH DISTRIBUTION

7.4.5 stress and largest crack

The relation between the stress and the size of the widest cluster, as illustrated by fig 7.12, is clearly a well defined power-law:

\[ \sigma(W) = a W^{-\beta}. \] (7.6)

We can derive \( \alpha \) and \( \beta \) both from direct data analysis or starting from the scaling parameters of the mean stress and the mean largest size. From the comparison of the so-calculated parameters with those directly obtained from the fit, we can have a guess of the stability of our constructions.

The calculated values of the parameters show a very good agreement with the fitted ones. From the scaling fits in figure 7.8 and 7.9, we obtained for the forms

\[ \sigma(L) = a [2 \ln(L) - n]^{-1/2}; \] (7.7)

\[ W(L) = A [M \ln(L) - N]^B, \] (7.8)

the parameters

\[ a = 0.9340, \quad n = 0.4421; \]

\[ A = 0.6239, \quad M = 1.4877, \quad B = 1.2971, \quad N = 0.3289. \]

If we invert \( W(L) \) and we insert \( L(W) \) back in \( \sigma(L) \), we obtain for the parameter of eq. (7.6):

\[ \alpha = \frac{a}{\sqrt{2}} \sqrt{MA^{1/B}}; \]

\[ \beta = 1/2B. \]
Figure 7.12: Mean stress as function of the largest size \( s(W) \). The parameter obtained by direct fitting are in good agreement with those calculated from the expression for the mean stress and largest size.

Calculated values are in good agreement with the fitted ones (see fig. 7.12):

\[
\begin{align*}
\text{calculated} & \quad \begin{cases} 
\alpha = 0.6716, \\
\beta = 0.3855
\end{cases} \\
\text{fitted} & \quad \begin{cases} 
\alpha = 0.6724, \\
\beta = 0.3863
\end{cases}
\end{align*}
\]

7.5 The Postfailure Regime

The postfailure regime is characterized, in the current-driven model, by the presence of one last single avalanche of bonds that break in sequence without the need of further increase in the current. To study the postfailure regime, we analyse the behaviour of the number of broken bond between the peak load and the complete failure.

How does the crack proceeds in the last avalanche? We can see if it advances just expanding itself, or including preexisting cracks.

The size of this last avalanche is simply given by the difference of the number of broken bonds between the peak load and the complete breakdown \((N_{la} = N_{last} - N_{C_{max}})\). Its scaling provide important informations about what is happening in this last period of time: if the fracturing cluster is simply expanding straightly across the lattice, we have \(N_{la} \sim O(L)\). If there are sparse damages in the meantime, we instead find \(N_{la} \sim O(\sqrt{L})\). In related, though slightly different, models this quantity is usually well represented by a power-law, with an exponent displaying weak topology dependence. It ranges between 0.61 and 0.81 for triangular, diamond and square lattices [54, 59, 93].
Here, the number of fuses broken in the last avalanche is

\[ N_{la} = N_{last} - N_{C_{max}} = 1.027L^{1.052}. \]  \hfill (7.9)

We obtain the number of assimilated clusters subtracting the number of clusters presents at the end from those presents at \( C_{\text{max}} \):

\[ M_{la} = M_{C_{\text{max}}} - M_{\text{last}} = 0.447L^{1.058}. \]  \hfill (7.10)

The difference between the two numbers is the number of bond “just appended” to the main crack:

\[ N_{\text{free}} = N_{la} - M_{la} = 0.543L^{1.060} \]  \hfill (7.11)

The number of broken bonds \( N' \) is \( O(L) \), then we can assume that

- almost all the bonds broken after the peak-load belong to the spanning cluster;
- crack branching or other path deviation are highly improbable.

that is, from the peak-load, the fracturing crack goes straight to percolation.

![Figure 7.13: Number of bonds broken in the last avalanche \( N_{la} \), compared to the number of cluster coalesced in the avalanche \( M_{la} \) (crack bridging). Their difference is the number of bonds \( N_{\text{free}} \) just appended to the advancing crack.](image)

In figure 7.14, we express the fraction of crack bridging (when a single broken bond make two cluster coalesce, a type II defect), with respect to the total number of broken bonds in the postfailure regime. This fraction grows up to \( L = 32 \) (maybe a finite-size effect), where it starts
to decrease (slowly than a logarithmic behaviour). From the data available we cannot know with certainty if this fraction will approach a constant or will go to zero, for very large $L$, even if we guess that the first hypothesis seems the most probable. Anyhow, this scaling sheds light on the propagation of the crack in the post-failure regime. Nearly half of the steps are done by bridging the spanning clusters with other smaller defects.

![Fraction of bridging steps in the postfailure regime. The fraction decrease is slower than a log.](image)
8.1 Diamondlike and Square Systems

We want to show here that there is an analogy between the square lattice model and the diamond lattice model. Note that the statistical data available for the square model are one tenth of those relative to the diamond model.

Density of broken bonds: In figure 8.1 we compare the density of broken bond at the peak load and at the complete breakdown in the square and diamond lattices. In the first case the data are almost superimposable; in the latter case the scaling is equivalent, but there is a difference in the prefactor that can be anyway related to the fact that the linear dimension $L$ of the diamond lattice in the periodic direction is double over the square lattice. From the comparison we can learn that the size of the last avalanche, in both cases, is a constant function of the linear size, that is in the postfailure regime the fracture propagates linearly towards percolation. The accumulated damage prior to the peak load is instead a subextensive function of the whole volume of the system, and does not depend (as far as we know) on the linear sizes.

Stress comparison: We display in figure 8.2 the comparison between the stress at the first bond breaking and at the peak load in the two topologies. The scaling behaviour seems equivalent in both cases, maybe a little better in the former one. A very interesting fact is that the two stress are interrelated by a single multiplicative constant, both in the first bond breaking and the peak load. This strong relationship is surprising given that the different topology shares the current in different ways: every bond of the diamond lattice connects two different rows, while in the square lattice the bonds are divided between parallels or perpendicular with respect to the direction in which the current flows.

The strength distribution: As expected from the behaviour of the stress, the scaling of the strength distribution is equivalent for the two lattices. Here one can recover the same behaviour of the diamond case: size-effects with shift correction and Gumbel distribution with the identical
Figure 8.1: Density of broken bonds at the peak load and during the postfailure.

Figure 8.2: Voltage values at 1bb and $C_{\text{max}}$ for diamondlike and square systems.
values for the parameters \( d = 2 \) and \( \mu = 2 \). The poor scaling of the tails in figure 8.4 is mainly
due to the poor statistics of the square model.

Figure 8.3: Mean stress at the peak load for square and diamond lattices.

Figure 8.4: Renormalization of \( S(\sigma) \). Square lattices.
8.2 Finite-Size Effects

The smaller sizes, in the voltage related variables, presents strong finite-size effects that affect the use of this variables.

The cause of this effects is an interesting phenomena that appears in all sizes smaller or equal to \( L = 32 \) and that suddenly disappears above this size. As shown in figure 8.3, sometimes happens that the fracturing cluster winds around the periodic direction of the lattice before closing, usually via branching. The principal effect observed is that \( N_{V_{\text{max}}} \) assumes a enormously large value than the average, while \( N_{C_{\text{max}}} \) remains unchanged. This yields, besides large values for the cluster sizes, anomalous values in the distribution of the voltages (and hence of the stresses).

In figure 8.4, we show the ratio between the number of fuse burned needed to reach the maximum voltage and the maximum current. While for larger size the two point coalesce, here we can have value of the ratio, for \( L = 32 \) up to 80 (out of scale in the figure). We believe, grounding on data analysis, that this phenomenon is a transient anomaly due to the small size of the system, rather than a stable size-effects of the model, supportes by the abrupt disappearance of the winding above a critical size. In the literature, as already mentioned in a previous chapter, one often found the usage of voltage related variables, such as \( V_{1}^{bb} \) and \( V_{\text{max}} \), in the study of the properties of the system. Specially the older works, due to the limitations in the then available computational power, rely just on this smaller sizes. We want to clearly point out that we discovered the existence of this phenomena only in the diluted disorder fuse random model and that we can't do any guess about their appearance in different, although related models.

Figure 8.5: A sample with crack windings.
We show the complete sequence of figures 8.8a-8.8h related to the distribution of the clusters at the various point (the beginning, $C_{\text{max}}$, $V_{\text{max}}$ and the breakdown) for all size to point out some interesting behaviours.

First it is clear the size range and the magnitude of the finite-size effect here discussed, broadening the low tail of the distribution for $L = 8$, 16, 32 and then disappeared. Another observation to be made is that, above $L = 128$, we can consider equivalent the data obtained at $V_{\text{max}}$ or
Finally the distribution of the sizes at the breakdown is narrower than that at peak load. This is expected because in the last avalanche following the peak nearly half of the broken bonds are bridging two clusters. The actual slope of this distribution presents anyway a size-dependent behaviour related to the main finite-size effect: for size below 32 its slope is narrower than that referring to the first bond breaking, while above this size its slope is broader.
Figure 8.8: Distribution of the widths $w$ of the clusters. Every single figure contains the data for all timesteps for a given $L$. 
By the means of the random fuse model we were able to probe the scaling properties of quasi-brittle porous material. Our results partly confirm the theoretical framework derived from the basic assumption of the theory of the linear elasticity, but at the same time we unveiled some new behaviour that need to be framed in a most general theory.

Some observations on our results are in order:

- we showed that there are some sort of mirrored properties at the very first bond breaking and at the peak load. At the beginning the largest crack hypothesis is valid, if referred not only to the largest defects, but to the set of two type of “most critical defects”, while the weakest-link hypothesis fails (cracks don’t percolate). On the opposite, at the peak load the latter is valid, since the defect that start here propagate to the percolation, while the other assumption is not valid, the defect that start the crack being not a critical one. The values of the critical exponent $\mu$ extracted from the study of the strength distribution at the two times confirms these facts and also point out that the stress enhancement behaviour at the peak is anyway dictated by the energetics of the largest cracks.

- the cluster defects distribution at the first bond breaking is, as expected, exponentially decaying and this yields a Gumbel distribution for the largest clusters. The subsequents modified Gumbel distribution for the first critical stress presents a value of the critical exponent $\mu = 1.4$, reflecting the energetic competition between the two different types of critical defects. We were also able to calculate a power-law shift-correction for the size-effects of the distribution.

- before the peak-load damage accumulate, modifying the distribution of the clusters. The new distribution results a bimodal exponential distribution that easily separates in a part relative to smaller cluster sizes, almost identical to the previous case, and a broader part relative to the larger sizes, that presents a nontrivial size dependence. This bimodal nature, being well-separated, does not affects the mode of the distribution of the largest defects, that remains unimodal; however we had to modify the form of the Gumbel distribution to
an expression similar to that used for the strength distribution to have a good scaling. This latter distribution is instead recovered in its usual form for the strength distribution at the peak load. A power-law shift correction for the size-effects was given here too, depending on the inverse of the size.

Besides, we want to point out some “side” results, that are, in our opinion, noteworthy at least for their meaning in the analysis.

• the “usual” expression for the mean stress is, at least in the large $L$ limit, the exact formula;

• there exist very strong finite-size effects, up to $L = 32$ included, due to the winding of one crack along the lattice and that this behaviour seriously affects the value of some voltage-related observables.
BIBLIOGRAPHY


Examiner Report for

Claudio Manzato

Failure of disordered media

dissertation submitted for the Degree of Doctor of Philosophy

The fracture of disordered media under different loading conditions is a very important scientific problem with a broad spectrum of technological applications. Investigations of the last three decades have revealed that the heterogeneous micro-structure of materials has a strong effect on fracture processes which can only be understood in the framework of statistical physics. The most noticeable consequence of the presence of disorder is the relatively low strength of materials compared to the value of the Young modulus, which additionally decreases with the sample size. Besides making materials weaker, disorder has also a very important "positive" effect in the fracture process: growing cracks can stop when reaching a material region with a higher local strength preventing catastrophic collapse. As a consequence, the macroscopic failure of heterogeneous materials is preceded by a damage accumulation process in the form of nucleation of new micro-cracks, furthermore, of the growth and arrest of the existing ones.

The thesis addresses interesting questions in one of the controversial issues of the research field, namely, the goal of the work is to investigate the size scaling of the strength distribution and to reveal the micro-structural background of the observed size effect. The work is purely theoretical, it is mainly based on computer simulations of the fuse model. In order to introduce disorder into regular fuse lattices the candidate applies random dilution, i.e. in the initial state a fraction \( n_r \) of fuses is removed in an uncorrelated fashion. The study is restricted to low values of \( n_r \) compared to the percolation threshold of the corresponding lattice type, however, simulations are performed using different lattice topologies. The goal of the work is clearly set in the thesis and the methods used in the investigations are adequate.
The thesis consists of an overview followed by the presentation of the candidate's own research. The overview presents the basic concepts, methods, and main results of the field of the fracture of disordered materials. This overview outlines some open questions of the field and serves to motivate the work presented in the thesis.

I think one of the main merits of the work is that in the framework of the thesis the scaling of the strength distribution and related size effects are investigated both by analytic calculations and by computer simulations. Scaling formulas are deduced which are then supported by the numerical evaluation of the simulated data. Another important point is that the scaling analysis is performed separately for the main regimes of the loading process, i.e. first bond breaking, damage accumulation up to the peak load, peak load, and the post failure regime. The high quality of the numerical data enables the author to make strong statements in favor of the Gumbel distribution.

The manuscript is well structured and clearly written. The presentation of the main ideas and results obtained is convincing. I have the following specific comments, suggestions and questions to improve the thesis:

1. The English should be considerably improved throughout the thesis. Careful polishing of the text is recommended before submitting the final version.

2. The goals of the thesis are first mentioned on Page 2 in a paragraph consisting of two sentences. Then the next time where the reader gets information about the subject of the thesis is on Page 45 after the complete review. I would recommend to extend a little bit the text on Page 2 giving a slightly longer, more detailed outline of the goals to satisfy the curiosity of the reader.

3. I think some of the equations are not correct due to mistyping: in Eq. (2.18) \( \mu \) seems to be missing; on the right hand side of Eq. (2.38) \( a \) should not occur; and in Eq. (4.12) probably \( l \) should be on the left hand side instead of \( n \).

4. In the inset of Fig. 5.4 no axis title is given which makes it difficult to understand the figure.

5. In order to make the meaning of peak current, maximum voltage, first bond breaking more transparent it could be useful to present the current-voltage diagram of a single sample with the appropriate notation. Figure 5.4 presents the time evolution of both the current and the voltage but the constitutive curve may still be helpful.

6. How is the width of cracks defined? Is it calculated as the usual radius of gyration?
7. Figure 5.3 demonstrates that the maximum current and maximum voltage becomes equal to each other at large enough system sizes. It seems to indicate that the response of the lattice becomes highly brittle in the limit of large sizes. It is known that when changing the mechanical response of a heterogeneous fracture model from quasi-brittle to brittle the size distribution of avalanches exhibits a crossover to a lower exponent. Does the result of Fig. 5.3 imply that increasing the system size such a crossover of the avalanche size distribution could be observed?

Summarizing, the thesis is based on a sound theoretical work, the results obtained provide an interesting contribution to the field of fracture of heterogeneous materials. The candidate should be granted the permission to defend his dissertation.